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ABSTRACT
This is unit four of a fifteen-unit SMSG secondary school text for high school students. The text is devoted almost entirely to mathematical concepts which all citizens should know in order to function satisfactorily in our society. Chapter topics include equations and inequalities and congruence. (MP)

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UNIT FOUR

Chapter 7. Equations and Inequalities

Chapter 8. Congruence

SE 027 899

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TABLE OF CONTENTS

Chapter 7

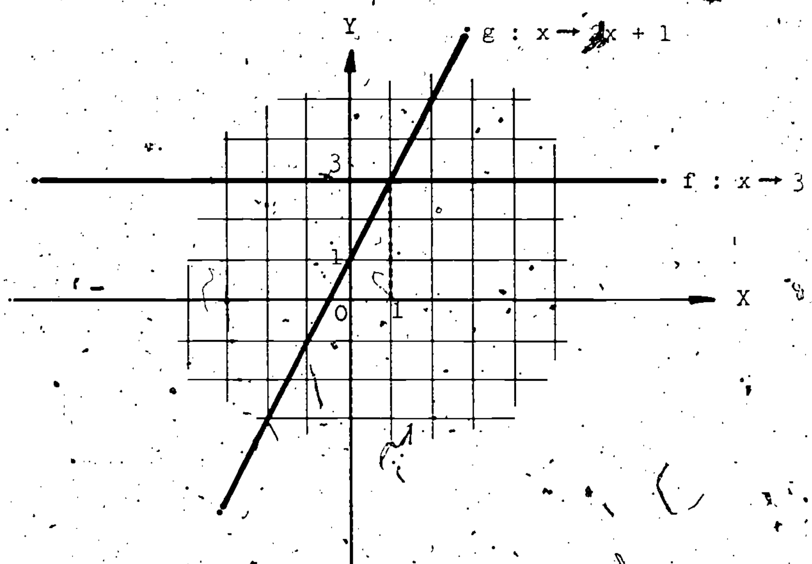
EQUATIONS AND INEQUALITIES

7-1. Solutions of Mathematical Sentences

In previous chapters you have graphed functions like

$$f : x \to 3, \text{ and}$$

$$g : x \to 2x + 1.$$



From the graph, it is clear that for an input of 1, the outputs of the functions f and g are exactly the same, namely 3. There are many real-life problems which lead to the question, "For what inputs, if any, are the outputs of two functions the same?" This situation is often represented by an equation (sometimes called a sentence), such as,

$$2x + 1 = 3.$$

However, the question still remains the same. For two given functions, is there an input such that the values of the functions are identical?

If we replace the variable by a number, then the equation becomes a statement which is either true or false, but not both. In most cases, it is

possible to decide whether a statement is true or false by doing some simple calculations, or by inspecting the given statement. In some unusual cases more complicated calculations or analyses are needed to determine whether the statement is true or false.

There are some equations where it is not possible to find any inputs that will result in a true statement. For example, consider the functions

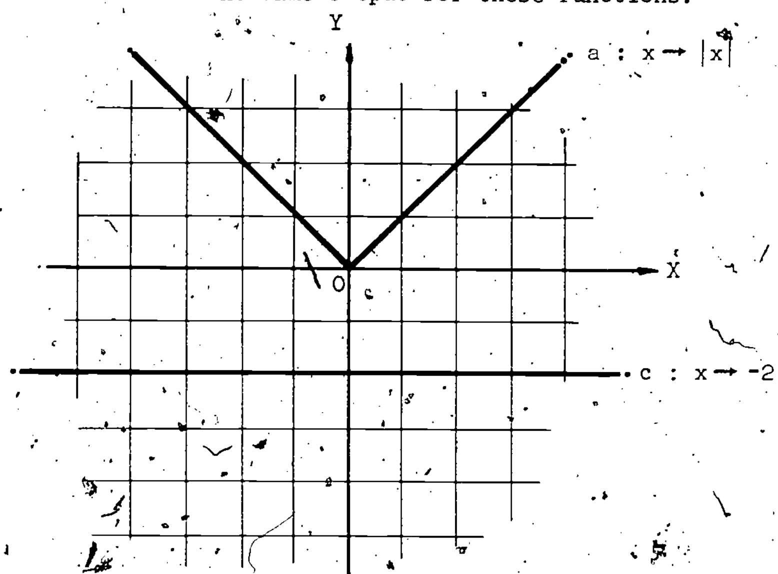
$$a : x \to |x|,$$

$$c : x \to -2,$$

and the equation,

$$|x| = -2.$$

It is easy to see from the graphs of the functions below that there is no input which will result in the same output for these functions.



There are also some equations where every acceptable input will result in a true statement. For example, consider the functions

$$h : x \to 2x + 6,$$

$$g : x \to 2(x + 3),$$

and the equation,

$$2x + 6 = 2(x + 3),$$

Since the graphs of the two functions, h and g, are identical, any acceptable input will result in the same output for each function.

We call the set of <u>inputs</u> which result in <u>true</u> statements the <u>solutions</u> or <u>the solution set</u> of the sentence. As you can see this set can have no members, one member, or more than one member. One of the purposes of this chapter is to develop some organized methods to find the solutions of some simple sentences in one variable.

<u>Exercises 7-1a</u>

(Class Discussion)

Which of the following statements are true? Which are false?

1. $4 + 8 = 10 + 5$
2. $8 + 3 = 10 + 1$
3. $4 + 8 = 8 + 4$
4. $5 + 7 \neq 6 + 6$ (The symbol "$\neq$" is read "is not equal to".)
5. $\frac{1}{2} + \frac{5}{8} = 1 + \frac{1}{8}$
6. $\frac{85}{1} \neq 85$
7. $13 + 0 \neq 15 + 0$
8. $12(5) \neq 5(12)$
9. $7(6 \times 3) = (7 \times 6) \times 3$
10. $8(\frac{1}{2} - \frac{1}{4}) = 8(\frac{1}{2}) - 8(\frac{1}{4})$
11. $65 \times 1 = 65$
12. $13 \times 0 = 13$
13. $\frac{2}{3}(7) \neq 2(\frac{7}{3})$
14. $4(\frac{3}{5}) = \frac{12}{5}$
15. $8(\frac{3}{5}) = \frac{24}{40}$

Exercises 7-1b

In each of the sentences in Exercises 1-8, determine whether replacing the variable(s) by the suggested value(s) results in a true statement or a false statement.

1. $7 + x = 12$; let x be 5

2. $7 + x \neq 12$; let x be 5

3. $y + 9 \neq 11$; let y be 6

4. $t + 9 = 11$; let t be 6

5. $\frac{5x + 1}{7} \neq 3$;

   (a) let x be 3

   (b) let x be 4

6. $2y + 5x = 23$;

   (a) let x be 4 and y be 3

   (b) let x be 3 and y be 4

7. $2a - 5 \neq (2a + 4) - b$;

   (a) let a be 9 and b be 9

   (b) let a be 3 and b be 9

8. $5m + x = (2m + 3) + x$; let x be 4

To the right of each sentence below is a set of inputs which contains all of the numbers belonging to its solution set, and <u>possibly</u> some that do not belong to the solution set. Find the solution set of each sentence.

9. $3(x + 5) = 17$; $\{0, \frac{1}{3}, \frac{2}{3}, 1\}$

10. $x^2 - (4x - 3) = 0$; $\{1,2,3,4\}$

11. $x^2 + \frac{1}{6} - \frac{7}{6}x = 0$; $\{1,2,6, \frac{1}{6}\}$

12. $x + \frac{1}{x} = 2$; $\{1,2,3\}$

13. $x(x + 1) = 3x$; $\{0,1,2\}$

14. $\frac{5x + 1}{7} = 3$; $\{0,2,4\}$

15. $x + 1 = 5x - 1$; $\{1, \frac{1}{2}, 2\}$

16. $x + 2 = x + 7$; $\{0,2,3\}$

7-2. A Systematic Method of Solution

Finding the solutions of sentences by graphical methods or by trial and error has some weaknesses. It is difficult to find the solution set of a given sentence by graphing if the solutions are not integers. It is also difficult to guess such a solution, except in very simple cases. For example, to guess that $\frac{11}{13}$ is the only solution of the equation

$$2(5x + 3) + 4 = 3(4 - x) + 9$$

would be difficult. We are not saying that the techniques of graphing or guessing are not useful. In fact, they can be the only methods available to us to find an approximation of the solutions of some equations. However, we need a more efficient way of finding the solution set of equations like the one above.

Consider, for example, the equation $2x - 3 = 7$. We first rewrite this equation as follows:

$$2x + (-3) = 7.$$

This sentence asserts that for some input the outputs of the two functions,

$$f : x \to 2x + (-3)$$

$$\text{and} \quad g : x \to 7,$$

are equal. Since the output of the constant function g is always 7, for any input, we are looking for an input, x, such that the value of the output of f, $2x + (-3)$, is 7.

To get the expression, $2x + (-3)$, we see that we started with a number, x,

$$\boxed{x}$$

multiplied by 2,

$$\boxed{x} \xrightarrow{\text{multiply by } 2} \boxed{2x}$$

and then added the opposite of 3.

$$\boxed{x} \xrightarrow{\text{multiply by } 2} \boxed{2x} \xrightarrow{\text{add } (-3)} \boxed{2x + (-3)}$$

Now the equation

$$2x + (-3) = 7$$

tells us that we want the value of the output, $2x + (-3)$, to be 7.

$$\begin{tikzcd}
\boxed{x} \arrow[r, bend left, "\text{multiply by } 2"] & \boxed{2x} \arrow[r, bend left, "\text{add } (-3)"] & \boxed{2x + (-3)} \\
 & & \boxed{7}
\end{tikzcd}$$

Reversing our steps we find the value of the output in the second box,

$$\begin{tikzcd}
\boxed{x} \arrow[r, bend left, "\text{multiply by } 2"] & \boxed{2x} \arrow[r, bend left, "\text{add } (-3)"] & \boxed{2x + (-3)} \\
 & \boxed{7 + 3} & \boxed{7} \arrow[l, bend left, "\text{add opp}(-3)"]
\end{tikzcd}$$

and then the value of the output in the first box.

$$\begin{tikzcd}
\boxed{x} \arrow[r, bend left, "\text{multiply by } 2"] & \boxed{2x} \arrow[r, bend left, "\text{add } (-3)"] & \boxed{2x + (-3)} \\
\boxed{\frac{1}{2}(7 + 3)} & \boxed{7 + 3} \arrow[l, bend left, "\text{multiply by the reciprocal of } 2"] & \boxed{7} \arrow[l, bend left, "\text{add opp}(-3)"]
\end{tikzcd}$$

We now list the successive outputs in equation form:

$$\begin{aligned}
2x + (-3) &= 7, \\
2x &= 7 + 3, \\
x &= \frac{1}{2}(7 + 3),
\end{aligned}$$

or in simplest form,

$$x = 5.$$

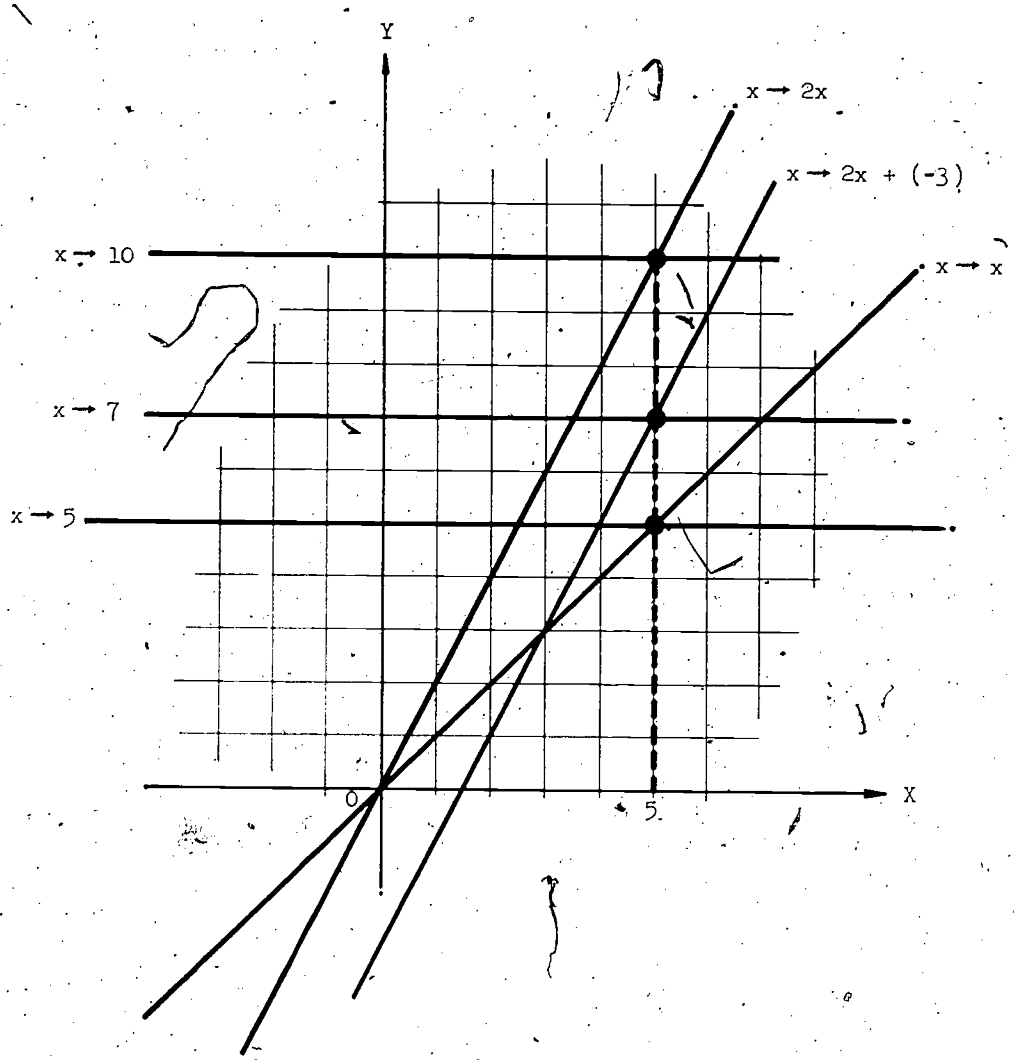
At each successive step we are asserting that for some <u>input</u> the values of the outputs of the pairs of functions,

(1) $x \to 2x + (-3)$ and $x \to 7$,

(2) $x \to 2x$ and $x \to 10$,

and (3) $x \to x$ and $x \to 5$,

are the same. <u>In fact the same input does this for each pair of functions</u> as shown in the graph below. Thus we are able to say that the solution of the sentence $x = 5$ is also the solution of $2x + (-3) = 7$. <u>Such equations having the same set of possible inputs and the same solution set are called equivalent equations.</u>



Exercises 7-2a

(Class Discussion)

Solve the following equations. In each case show the appropriate boxes. Finally, express each solution in simplest form.

1. $2x + 7 = 15$
2. $5(x - 3) = 2$
3. $\frac{4}{3}(x + \frac{6}{5}) = 8$
4. $\frac{2x + 5}{6} = 3$

The method used to solve the above equations, although limited in the kinds of equations it can deal with, does identify some of the basic procedures we want to develop in our more systematic and useful approach. Namely:

(1) it indicates that <u>addition</u> and <u>multiplication</u> are <u>basic</u> to simplifying the expressions of an equation and suggests an acceptable order for using them;

(2) it emphasizes the fact that our goal is to find a simple equation of the form $x = c$ where the solution set is easily recognized.

The boxes used in the preceding exercises were introduced as an aid to learning. Now we will learn how to solve equations without the use of the boxes.

Consider the sentence

$$5x + (-2) = 13.$$

If we use 3 for the input x, the values of the outputs of the two functions

$$f : x \rightarrow 5x + (-2)$$

and

$$g : x \rightarrow 13$$

are $5(3) + (-2)$ and 13,

and the sentence becomes a <u>true statement</u>

$$5(3) + (-2) = 13$$

or

$$13 = 13.$$

That is if x is replaced by a solution of the equation, then "$5x + (-2)$" and "13" are two different names for the same number. If

x is 3, we can multiply by 10 and get

"$(10)(5x + (-2))$" and "$(10)(13)$"

which are two different names for the same number, 130. If x is 3, then we can add 30 and get

"$5x + (-2) + (30)$" and "$13 + (30)$"

which are two different names for the same number, 43. This situation will occur only when the number replacing x is a solution of the given equation, in this case $5x + (-2) = 13$.

Exercises 7-2b

(Class Discussion)

The first statement in each exercise is a true statement.

1. $(10 + 5) = 15$. Are "$(9)(10 + 5)$" and "$(9)(15)$" two different names for the same number?

2. $\frac{52}{4} = 12 + 1$. Are "$2 + \frac{52}{4}$" and "$2 + (12 + 1)$" two different names for the same number?

3. $4,721,652 = (3)(1,573,884)$. Are "$4,721,652 + 5,649,721$" and "$(3)(1,573,884) + 5,649,721$" two different names for the same number?

4. $(4,721,652 - 2,510,342) = 2,211,310$. Are "$(3,284,621)(4,721,652 - 2,510,342)$" and "$(3,284,621)(2,211,310)$" two different names for the same number?

In the process of solving equations using the boxes you saw that you could use the operations of addition or multiplication to simplify the expressions in the equation. The best part of that process was being able to see what steps to take to undo or simplify the expressions in the equation. Basically you looked at an expression like $2x + 10$, decided what steps you would take, starting with x, to construct this expression, and then you reversed the steps to simplify it. Now let's try to do this without the boxes and develop a method that will solve many more equations.

Example: If there is an input such that

$$x = 3$$

is a true statement, then does the same input also make the following equation a true statement:

$$5x + 2 = 17?$$

(1) Start with $\qquad x = 3.$

(2) Multiply by 5. $\qquad (5)x = (5)3$

(3) Add 2. $\qquad 5x + 2 = 15 + 2$

$\qquad$ or $\quad 5x + 2 = 17.$

(4) Now start with $\qquad 5x + 2 = 17.$

(5) Add (-2). $\qquad 5x + 2 + (-2) = 17 + (-2)$

$\qquad$ or $\quad 5x = 15.$

(6) Multiply by $\frac{1}{5}$. $\qquad (\frac{1}{5})(5x) = (\frac{1}{5})(15)$

$\qquad$ or $\quad x = 3.$

Exercises 7-2c

(Class Discussion)

1. Starting with $x = 5$ show how you can get $2x + 3 = 13$. Reverse the process and show how to get $x = 5$ starting with $2x + 3 = 13$.

2. Starting with $x = 6$ show how you can get $3(x - 7) = -3$. Reverse the process and show how you can get $x = 6$ starting with $3(x - 7) = -3$.

3. Starting with $x = \frac{7}{5}$, show how you can get $5x + 2 = 9$. Reverse the process and show how you can get $x = \frac{7}{5}$ starting with $5x + 2 = 9$.

4. Starting with $x = -2$, show how you can get $5 - 3x = 11$. Reverse the process and show how you can get $x = -2$ starting with $5 - 3x = 11$.

We now have the basic parts of our more systematic method of finding the solution set of an equation. We will summarize the procedure that has been developed thus far.

(1) If a sentence, like $2x + 3 = 17$, becomes a true statement for some input x, then we can add any number to the number on both sides of the sentence and the resulting sentence will be a true statement for the <u>same</u> input x.

(2) If a sentence, like $2x + 3 = 17$, becomes a true statement for some input x, then we can multiply the number on <u>both</u> sides of the sentence by any <u>nonzero</u> number and the resulting sentence will be a true statement for the same input x.

(3) For each step in the simplification we select one of the operations, addition or multiplication, and a number such that an indicated operation in the sentence is undone.

(4) If the operation to be undone is the addition of some number, then you add the opposite of that number to the number on <u>both</u> sides of the equation.

(5) If the operation to be undone is multiplication by some number, then you multiply the number on <u>both</u> sides of the equation by the reciprocal of the multiplier.

Now our work could be arranged as follows:

If $2x + 3 = 17$ is a true statement for <u>some</u> input x,

then $2x + 3 + (-3) = 17 + (-3)$ is a true statement for the <u>same</u> input x,

$2x + 0 = 14$ is a true statement for the <u>same</u> input x,

$(\frac{1}{2})2x = (\frac{1}{2})(14)$ is a true statement for the <u>same</u> input x,

$x = 7$ is a true statement for the <u>same</u> input x.

The solution set of the last equation is $\{7\}$. Since each step is reversible, $\{7\}$ is also the solution set of the first equation. We know that each step is reversible since every number, except zero, has a reciprocal, and every number has an opposite.

Exercises 7-2d

Find the solution sets of the following equations. Show your work and be sure that each step is reversible.

1. $4y + 5 = 45$
2. $12 - y = 8$
3. $3x - 2 = 10$
4. $n - 5 = 7$
5. $4n - 5 = 7$
6. $12n - 5 = 7$
7. $x + \frac{3}{2} = 10$
8. $y - \frac{3}{2} = \frac{5}{2}$
9. $\frac{13}{7} = 1 + x$
10. $x + (-\frac{4}{9}) = (-\frac{7}{13})$
11. $-4y - 3 = \frac{9}{2}$
12. $.08d = 73$
13. $\frac{1}{2}x = 17$
14. $6 = \frac{x}{18}$
15. $6.2 + x = 1.12$

7-3. Simplifying

Sometimes we are asked to find the solution of an equation like

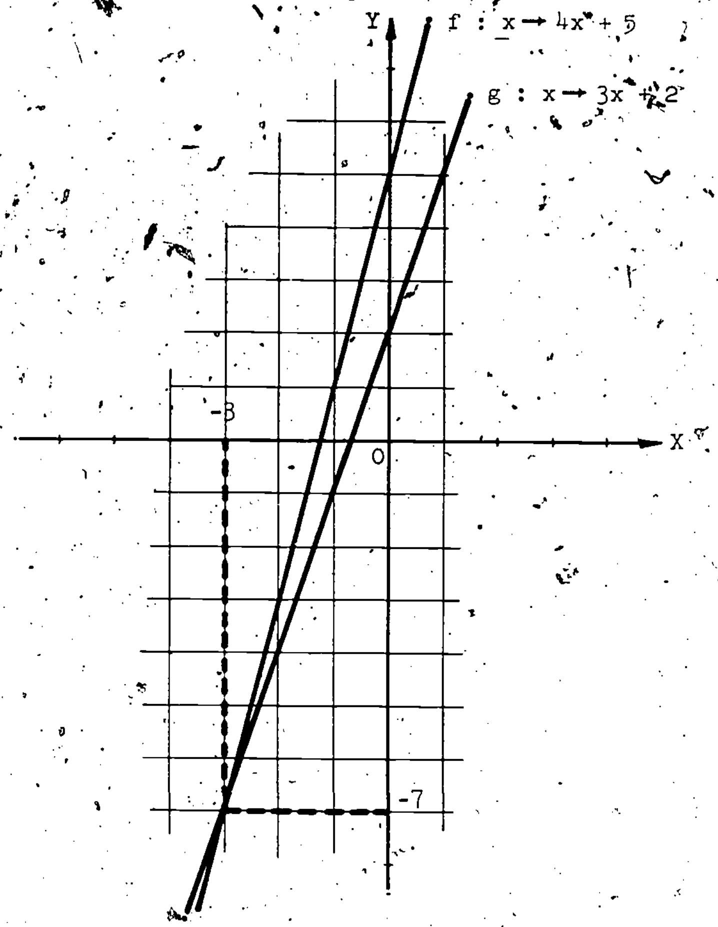
$$4x + 5 = 3x + 2.$$

In other words, we are looking for an input x such that the outputs of the functions

$$f : x \to 4x + 5$$

$$\text{and } g : x \to 3x + 2$$

are equal. From the graph below we see that the outputs of the two functions are equal for an input of -3.



We can also use our systematic approach to find the solution set of such equations.

<u>If</u> $4x + 5 = 3x + 2$ is a true statement for <u>some</u> input x,
then $4x + (-3x) + 5 = 3x + (-3x) + 2$ is a true statement for the same input x,
$x + 5 = 2$ is a true statement for the same input x,
$x + 5 + (-5) = 2 + (-5)$ is a true statement for the same input x,
$x = -3$ is a true statement for the same input x.

The solution set of this last sentence is $\{-3\}$. Since each step is reversible, the solution set of the original sentence is $\{-3\}$.

Consider the sentence

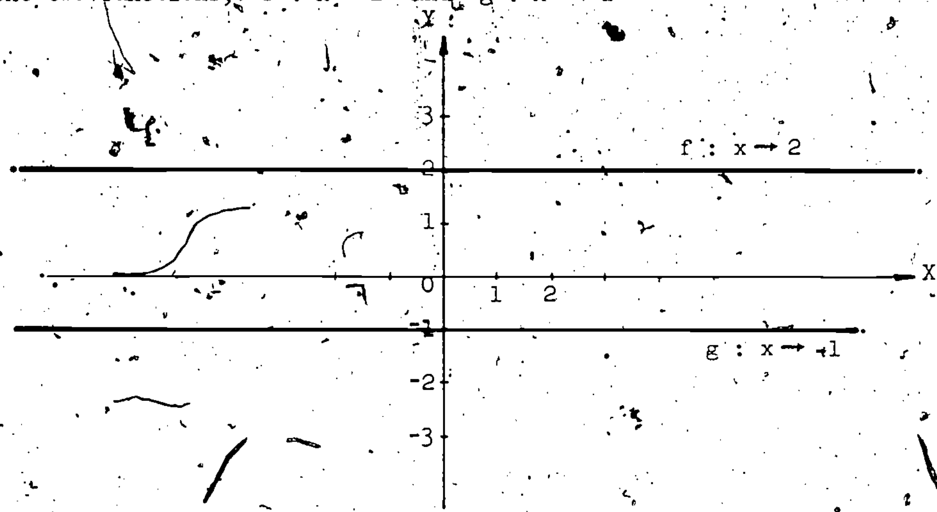
$$2 + 3x = 3x - 1.$$

Let's find its solution set.

<u>If</u> $2 + 3x = 3x - 1$ is a true statement for <u>some</u> input x,
then $2 + 3x + (-3x) = 3x + (-3x) - 1$ is a true statement for the <u>same</u> input x,

$2 = -1$ is a true statement for the same input x.

Obviously there is no input that will make this statement true. The graphs of the two functions, $f : x \to 2$ and $g : x \to -1$



also verify that no input exists that will result in equal outputs for these two functions. In this case we say that the solution set of the original sentence is the empty set, $\emptyset$.

Exercises 7-3a
(Class Discussion)

Simplify each of the following:

1. $2x + 3x$
2. $(-2)x + 3x$
3. $-x + 7x$
4. $-x + (-2x)$
5. $5x + (-5x)$
6. $(-2x) + (-5x)$

Exercises 7-3b

Find the solution sets of the following equations.

1. $2x + 3 = x + 2$ (Use the graphical method and then solve algebraically.)
2. $2x + 3x + 5 = 9$
3. $2x + 3(x + 5) = 9$

4. $2x + 7 = x$

5. $x = 2x + 6$

6. $3x + 5 = 2x$

7. $12x + (-6) = 7x + 24$

8. $8x + (-3x) + 2 = 7x + 8$

9. $5y + 8 = 7y + 3 + (-2y) + 5$

10. $2y + (-6) + 7y = 8 + (-9y) + (-10) + 18y$

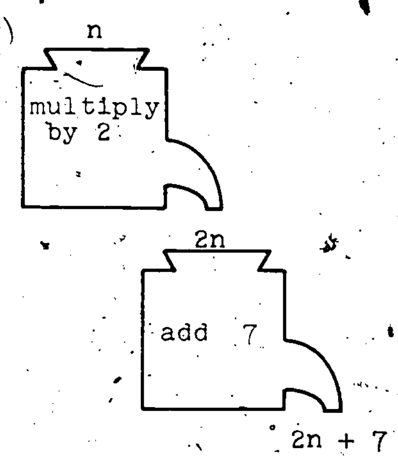
7-4. English Sentences and Mathematical Sentences

Mathematical sentences are used in many ways in many different fields. Equations serve as models of electrical circuits, weather patterns, and to describe what is happening in a cancer cell. One of the important steps in constructing mathematical sentences, that can serve as models of real life situations, is the translation of English sentences into mathematical sentences.

A first step is to learn to translate English phrases into mathematical phrases. For example:

> Write a mathematical phrase that can serve as a model of the English phrase, "The sum of twice a number and seven".

You can think of the translation of mathematical phrases as a series of function machines that are hooked together in the following way. Put a number n in the first machine. Then let the output of the first machine become the input of the second machine, and so forth.



Exercises 7-4a

(Class Discussion)

Write mathematical phrases representing each of the following.

1. A number added to four.
2. A number increased by seven.
3. Five subtracted from a number.
4. A number subtracted from five.
5. The product of nine and a number.
6. A number subtracted from twice the number.
7. The quotient of three times a number and 2.
8. Increase two times a number by 3.
9. Ten less than seven times a number.
10. The amount represented by the quotient of eight times a number and 5.

When we write mathematical sentences that are models of English sentences we usually are translating sentences that say, for some set of inputs the output of one function is equal to, unequal to, less than, or greater than, the output of a second function. The verb forms in these sentences are represented by the symbols $=$, $\neq$, $<$, and $>$.

Example: Consider the problem, "The sum of a certain number and eight is equal to two more than the product of four and the number. What is the number?" If we let x represent a number, then we can identify two functions in the above statement;

$$f : x \to x + 8 \quad \text{and} \quad g : x \to 4x + 2.$$

The problem says that for some input x the outputs of these two functions are equal. This situation is represented by the equation

$$x + 8 = 4x + 2.$$

Solution:

$$\begin{aligned}
x + 8 &= 4x + 2 \\
x + 8 + (-8) &= 4x + 2 + (-8) \\
x &= 4x - 6 \\
(-4x) + x &= (-4x) + 4x - 6 \\
-3x &= -6 \\
(-\tfrac{1}{3})(-3x) &= (-\tfrac{1}{3})(-6) \\
x &= 2
\end{aligned}$$

Exercises 7-4b

For each of the following problems write a mathematical sentence which can serve as a model of the situation. Then find the solution set of each sentence and give the answer to the question.

1. If one is added to twice a boy's age the result is nineteen. What is the boy's age?

2. A number is increased by 7 and the sum is tripled. The result is decreased by 5 and the difference is divided by 4. The final result is 8. What is the original number?

3. One number is three times as large as another number. Their difference is 12. What is the smaller number?

4. A board 45 inches long is to be cut into two pieces so that one piece is 3 inches longer than the other. Find the length of each piece.

5. The sum of a number and twice that number is 27. What is the number?

6. Take a number. Add 5 to it and multiply the result by 4. If your result is 48, then what was the original number?

7. The sum of three numbers is 83. The first number is double the second and the third number is 7 more than the second. Find the three numbers.

8. Two cars start from the same point at the same time and travel in the same direction at constant speeds of 34 and 45 miles per hour, respectively. In how many hours will they be 35 miles apart?

9. A student has test grades of 75 and 82. What must he score on a third test to have an average of 87?

10. Mr. Jones promised that when the family went to Southern California in the summer, Johnny could spend 5 days all by himself in Disneyland, Knott's Berry Farm and Marineland. Johnny saved up the money to spend on the trip. On each of the five days, in the morning before Johnny set out, Mr. Jones added $5 to the amount Johnny already had. Each day Johnny limited himself to spending half the amount he had when he set out in the morning. At the end of the five days Johnny found that he had exactly $4.85 left. How much was it that Johnny had saved up to spend on the trip?

7-5. Inequalities

In many mathematical sentences the order symbols, $<$ or $>$, and the symbol $\neq$ appear. Some examples of inequalities are

$$3x - 7 < 8,$$
$$2x > 4,$$
$$\text{and}\quad x + 2 \neq 3.$$

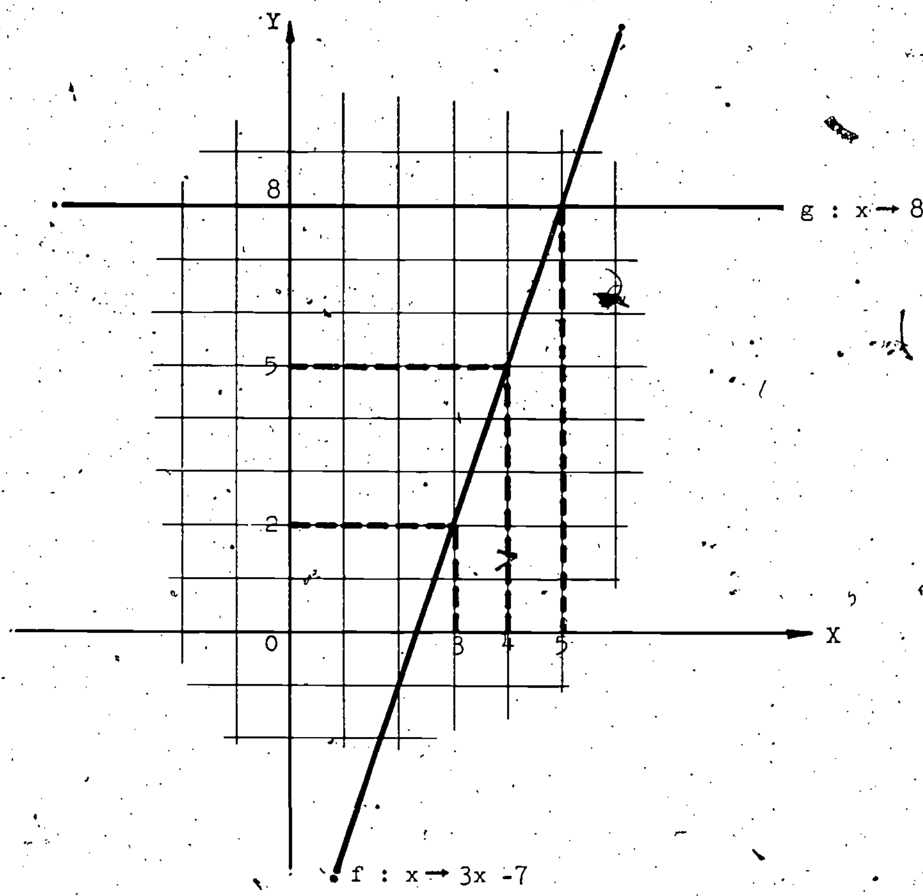
These are read "3x minus 7 is less than 8",
"2x is greater than 4,
and "x plus 2 is not equal to 3".

We will find that the sentence $3x - 7 < 8$ will become a true statement for some values of the variable x and a false statement for others. The values for which the sentence becomes a true statement are called the solutions of the inequality. Thus, if the input x is replaced by 3 in the function

$$x \rightarrow 3x - 7,$$

then the output becomes $3(3) - 7$ which is 2. Since 2 is less than 8, the sentence $3x - 7 < 8$ becomes a true statement when x is replaced by 3. Hence 3 is a solution of the inequality. To find some others let's look at the graph of the two functions

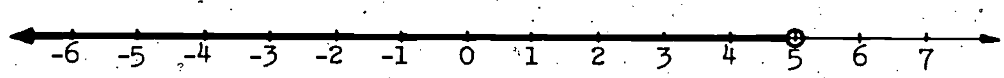
$$f : x \rightarrow 3x - 7 \quad\text{and}\quad g : x \rightarrow 8.$$



It is clear that the values of the two functions are equal for an input of 5. It should be clear, from the graph, that only for an input less than 5 is the output of f: x → 3x - 7 less than the output of g : x → 8. Furthermore for every number less than 5, the output of f is less than the output of g. Therefore we say that the solution set of the sentence

$$3x - 7 < 8$$

is the set of all numbers less than 5. We can represent this set on the number line as follows:

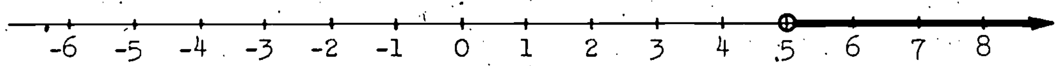


The open circle means that 5 is not included in the solution set.

Since the output of f : x → 3x - 7 is greater than the output of g : x → 8 for inputs greater than 5, and for no other values of x, the solution set of the sentence

$$3x - 7 > 8$$

is the set of all numbers greater than 5. This set can be shown on the number line as follows:



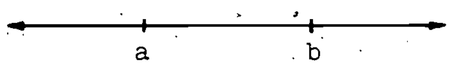
<u>Exercises 7-5a</u>

(Class Discussion)

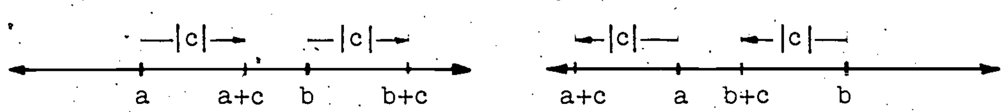
Using a graph, determine the solution sets of the following mathematical sentences. Represent the graphs of the solution sets on the number line.

1. $2x - 2 = -3x + 3$
2. $2x - 2 < -3x + 3$
3. $2x - 2 > -3x + 3$

It is possible to find the solution sets of inequalities in a manner similar to the method we developed for equations. To do this we need to recognize two properties of order. Let us fix two points, a and b, on the number line with $a < b$.



If we add the same number <u>c</u> to <u>a</u> and to <u>b</u>, then we move to the right $|c|$ units if <u>c</u> is <u>positive</u>, and to the left $|c|$ units if <u>c</u> is <u>negative</u>.



In both cases $a + c$ is to the left of $b + c$. We state the following property of order.

> <u>Additive Property of Order</u>. If a, b, c, are rational numbers and if $a < b$, then $a + c < b + c$.

Exercises 7-5b

(Class Discussion)

Apply the additive property of order to determine which of the following statements are true.

1. $(-\frac{6}{5}) + 4 < (-\frac{3}{4}) + 4$
2. $(-\frac{5}{2}) + (-5) < (-\frac{5}{3})(\frac{6}{5}) + (-5)$
3. $(-5.3) + (-2)(-\frac{4}{3}) < (-0.4) + \frac{8}{3}$
4. Illustrate the additive property of order, using the number line, for $a = -3$, $b = 1$ with c having successively the values -2, $\frac{5}{2}$, 0, 3.

We can use the additive property of order to find the solution sets of inequalities in the following way.

Example: Find the solution set of $x + 3 < 1$.

If $x + 3 < 1$ is a true statement for some number x,
then $x + 3 + (-3) < 1 + (-3)$ is a true statement for the same number x,
$x < -2$ is a true statement for the same number x.
Since these steps are reversible the solution set of the first sentence is the same as the solution set of the last sentence, namely,

the set of all numbers less than -2.

Exercises 7-5c

Find the solution sets of the following sentences. Graph the solution sets on the number line.

1. $3 + x < (-4)$
2. $(-3) < x + (-2)$
3. $2x < (-5) + x$
4. $3x > \frac{4}{3} + 2x$
5. $(-\frac{2}{3}) + 2x > \frac{5}{3} + x$
6. $(-2) + 2x < (-3) + 3x + 5$
7. $(-\frac{3}{4}) + (\frac{5}{4}) > x + (\frac{3}{2})$

Consider the true statement

$$5 < 8.$$

If each of these numbers is multiplied by 2, the inequality

$$(5)(2) < (8)(2),$$

or $$10 < 16,$$

is also a true statement.

Exercises 7-5d

(Class Discussion)

Complete Exercises 1 - 6 by inserting an inequality symbol "<" or ">" to make a true statement.

1. (a) 7 ______ 10
   (b) $6(7)$ ______ $6(10)$
2. (a) -9 ______ 6
   (b) $5(-9)$ ______ $5(6)$
3. (a) 2 ______ 3
   (b) $-4(2)$ ______ $-4(3)$
4. (a) -7 ______ -2
   (b) $2(-7)$ ______ $2(-2)$
5. (a) -1 ______ 8
   (b) $-3(-1)$ ______ $-3(8)$
6. (a) -5 ______ -4
   (b) $-6(-5)$ ______ $-6(-4)$
7. Look at the two parts of each exercise.
   (a) In which exercises did you use the same inequality symbol in (a) and (b)?
   (b) In which exercises did you use different symbols in (a) and (b)?
   (c) Choose one example in which you used the same inequality symbol and one in which you used different symbols. Can you see what caused the difference?

The above exercises suggest another important property of order.

<u>Multiplicative Property of Order</u>. If a, b, and c are rational numbers and if $a < b$, then

$$ac < bc, \text{ if } c \text{ is positive;}$$
$$bc < ac, \text{ if } c \text{ is negative.}$$

We can use both of the properties of order to find the solution sets of sentences like

$$6x - 45 < 105.$$

If $6x + (-45) < 105$ is a true statement for some number x,

then $6x + (-45) + (45) < 105 + (45)$ is a true statement for the same number x,

$6x < 150$ is a true statement for the same number x,

$(\frac{1}{6})\ 6x < (\frac{1}{6})\ 150$ is a true statement for the same number x,

$x < 25$ is a true statement for the same number x.

Since the steps are reversible the solution set of the sentence

$$6x - 45 < 105$$

is the set of all numbers less than 25.

Exercises 7-5e

Find the solution sets of the following sentences.

1. $2x + 3 < 9$
2. $5x - 8 < 10$
3. $7 - 3x > 2$
4. $(-2x) + 3 < -5$
5. $(-2) + (-4x) > (-6)$
6. $(-4) + 7 < (-2x) + (-5)$
7. $5 + (-2x) < 4x + (-3)$
8. $5x + 3x - 9 < 4$
9. $(\frac{2}{3}) + (-\frac{5}{6}) < (-\frac{1}{6}) + (-3x)$
10. $2x + 5 + (-4)(x + 3) < 6$

Translate the following situations into mathematical sentences and solve.

11. Johnny Jones had a bunch of dimes and one half-dollar. In all he had more than two dollars. How many dimes must he have had?

12. Johnny is 2 years older than Sally. The sum of their ages is less than their father's age which is 42 years. How old is Sally?

13. If Johnny is 2 years older than Sally and the sum of their ages is greater than their mother's age which is 39 years, how old is Sally?

14. The lines in Miss Jones' grade book for Richard, Tom and Ann look like this:

	First Test	Second Test	Third Test
Richard	92	77	?
Tom	81	56	?
Ann	78	67	?

Each of the students would like to have an average greater than 80 after the third test. For each student find what he must score on the third test so as to obtain an average greater than 80 if this is possible. (The maximum score on any test is 100.)

15. Dave Hart decides to spend the summer vacation earning money for his college education. He figures that there are 49 working days and that his expenses will amount to $69. How much does he have to earn per day in order to save a total greater than $470?

16. Mr. Robinson drives a compact car. He finds that he can get 22 miles per gallon of gasoline on the open road and 16 miles per gallon of gasoline in heavy traffic. On a trip of over 142 miles Mr. Robinson used 7 gallons of gasoline. How many miles of the trip were on the open road? (<u>Hint</u>: Let x represent a number of gallons of gasoline a car used on the open road.)

17. Mrs. Tobin has a large family. She planned a family party at which she served turkey and ham. She ordered a twelve-pound turkey and an eight-pound ham. If ham costs 15 cents per pound more than turkey and her bill was less than $10.20, what is the cost of a pound of turkey?

7-6. Approximating Solutions

We can also use the graphical method to find solutions of a new kind of equation though it will usually give only approximate solutions. Consider the following situation:

> "Assume that an object thrown vertically downward from the top of a cliff above a lake falls according to the law $16x^2 + 48x = s$ where s represents the distance that the object falls during the first x seconds. How long does it take for the object to travel the 64 feet to the surface of the lake?"

The model of the situation is

$$16x^2 + 48x = 64.$$

Multiplying by $\frac{1}{16}$ we get

$$x^2 + 3x = 4.$$

Adding the opposite of $3x$, we get

$$x^2 = -3x + 4.$$

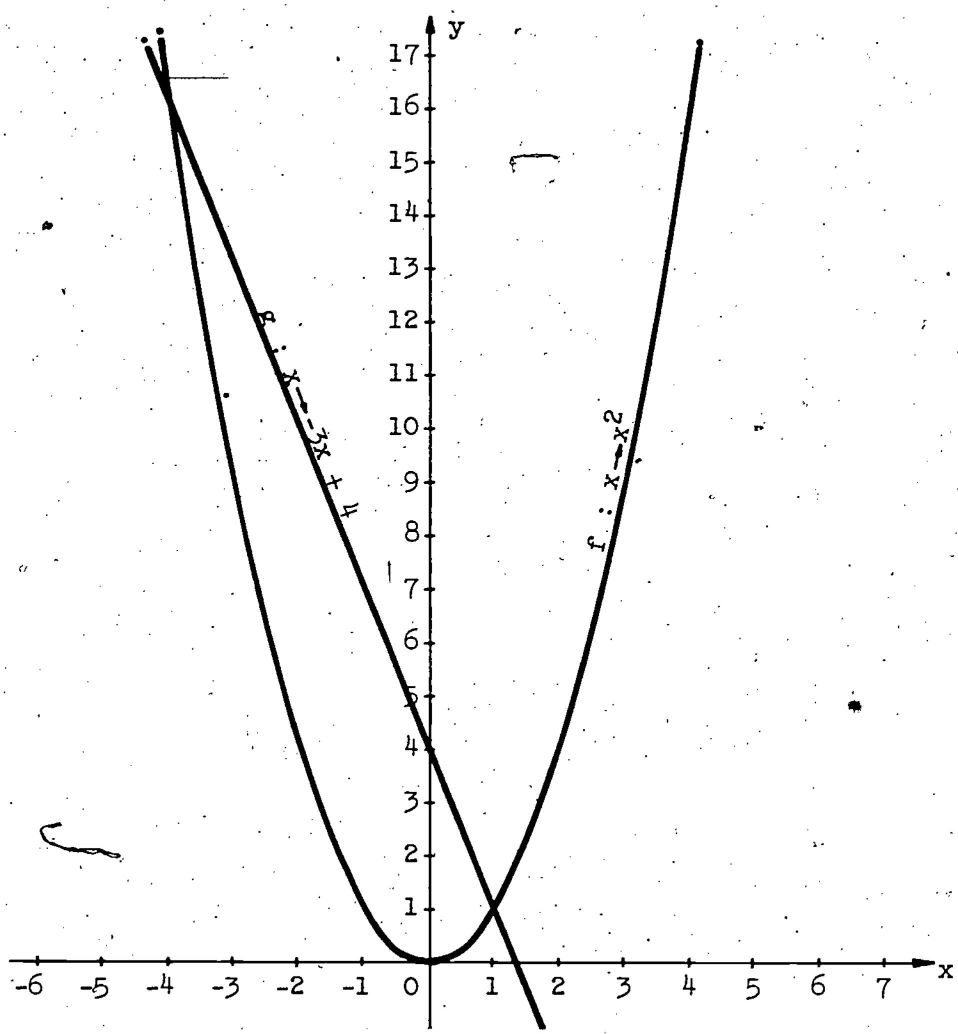
Here we graph the two functions

$$f : x \to x^2 \quad \text{and} \quad g : x \to -3x + 4.$$

The graph of g is just a straight line. The graph of f, however, is curved and we will need to plot a number of points in order to sketch the graph with reasonable accuracy.

x	0	1	2	3	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	4	
x^2	0	1	4	9	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{9}{4}$	$\frac{25}{4}$	$\frac{49}{4}$	16	

Noting that the square of a number is the same as the square of its opposite (for example, $3^2 = 9$ and $(-3)^2 = 9$) we now sketch the graph.



We note that there are two inputs for which the two functions have equal outputs, namely 1 and (-4). The only result that makes sense when the solutions from the mathematical model are interpreted in the problem situation is 1. That is, the object will hit the lake surface after 1 second.

We can also see from the graph that the solution set of

$$x^2 < -3x + 4$$

is the set of all numbers less than 1 and greater than -4, since the graph of f is below the graph of g in this interval. The solution set of

$$x^2 > -3x + 4$$

consists of those numbers, x, for which either

$$x < -4 \text{ or } x > 1.$$

If these points did not have integral coordinates, then we could estimate the values of the inputs from the graph.

Exercises 7-6

1. Use the graphical method to find approximate solution sets for each of the following.

$$2 - \frac{3}{4}x = 5$$

$$2 - \frac{3}{4}x < 5$$

$$2 - \frac{3}{4}x > 5$$

2. Use the graphical method to find approximate solutions sets for each of the following.

$$\frac{2}{3}x - 3 = 2 - \frac{3}{4}x$$

$$\frac{2}{3}x - 3 < 2 - \frac{3}{4}x$$

$$\frac{2}{3}x - 3 > 2 - \frac{3}{4}x$$

3. Use the graphical method to find approximate solutions for each of the following.

$$x^2 = \frac{1}{2}x + 3$$

$$x^2 < \frac{1}{2}x + 3$$

$$x^2 > \frac{1}{2}x + 3$$

7-7. Summary

Section 7-1.

An equation may be thought of as asserting that for some input the values of the outputs of two functions are equal.

If a value is assigned to the variable in an equation in one variable, then the equation becomes a statement which is either true or false, but not both.

We call the set of values that are assigned to the variable in

an equation, and that result in a true statement, the <u>solution set</u> of the equation.

Section 7-2.

We usually "solve" equations by finding simpler equivalent equations.

<u>Equivalent equations</u> are equations that have the same set of possible inputs for the variables and the same solution set.

We often use addition of opposites and multiplication by reciprocals to find simpler equivalent equations.

The statement $a = b$ means that "a" and "b" are different names for the same number. If "a" and "b" are names for the same number, then "a + c" and "b + c" similarly are two names for a single number, as are "ac" and "bc". Thus, if $a = b$, $a + c = b + c$ and $ac = bc$. It is also the case

(1) if $a + c = b + c$, then $a = b$

and (2) if $ac = bc$ and $c \neq 0$, then $a = b$.

Section 7-3.

When the variable occurs in more than one term in an equation, the equation can be simplified in one of two ways:

(1) by combining the terms using the distributive property if they are on the same side of the equal sign,

or (2) by adding the opposite of one of the numbers to the numbers on both sides of the equation.

Section 7-4.

One of the important steps in finding a mathematical model of a real-life situation is the translation of the verbal statement of the situation into meaningful mathematical symbols. The variables should be described carefully. The description of the variable should say what it measures, such as whether it is the number of inches, the number of donkeys, or the number of tons of wheat.

Each problem has three distinct parts in its solution:

(1) the translation, or the creation of the mathematical model of the situation,

(2) the solution of the mathematical sentence or sentences,

and (3) the interpretation of the solution in the original situation.

Section 7-5.

It is customary to call a simple mathematical sentence involving "<" or ">" an inequality. The solution sets of inequalities usually are sets with many numbers in them and it is sometimes convenient to represent these sets on the number line.

There are two useful properties of order that we use to simplify inequalities. They are:

Additive Property of Order: For any rational numbers a, b, and c, if $a < b$, then $a + c < b + c$.

Multiplicative Property of Order: For any rational numbers a, b, and c, if $a < b$ and $0 < c$, then $ac < bc$; if $a < b$ and $c < 0$, then $bc < ac$.

Section 7-6.

The approximate solutions of sentences like $x^2 = ax + b$, $x^2 < ax + b$, and $x^2 > ax + b$ can quite often be found by using graphical methods. To do this we graph the functions

$$f : x \to x^2 \quad \text{and} \quad g : x \to ax + b$$

and look at the values of their outputs for different inputs.

Chapter 8

CONGRUENCE

8-1. A Road Building Problem

A story is told that in ancient Greece there were two towns separated by a mountain. It was desired to build a straight road joining these towns by tunneling through the mountain.

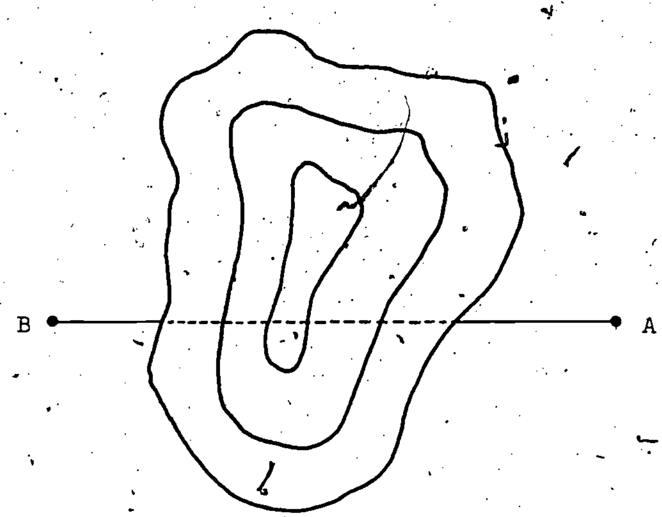


Figure 1

The towns, A and B, and the mountain are shown in Figure 1.

The ancient Greeks used their knowledge of geometry to solve this problem in the following way. First they picked a point C from which both towns A and B were visible. Then line segments were drawn connecting C with A and B.

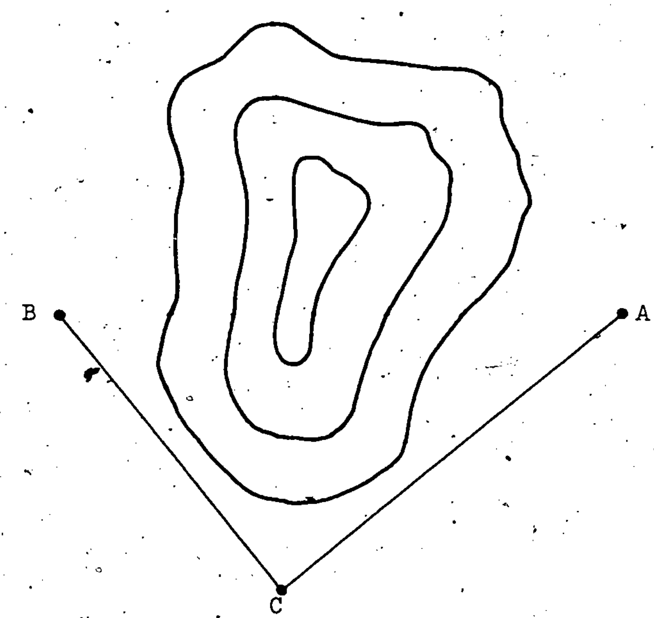


Figure 2

35

On the line $\overleftrightarrow{AC}$ they located point D so that the distances from C to D and from C to A were the same. Similarly they located point E on the line $\overleftrightarrow{BC}$ so that the distances from C to E and from C to B were equal. The segment $\overline{DE}$ was drawn as shown below.

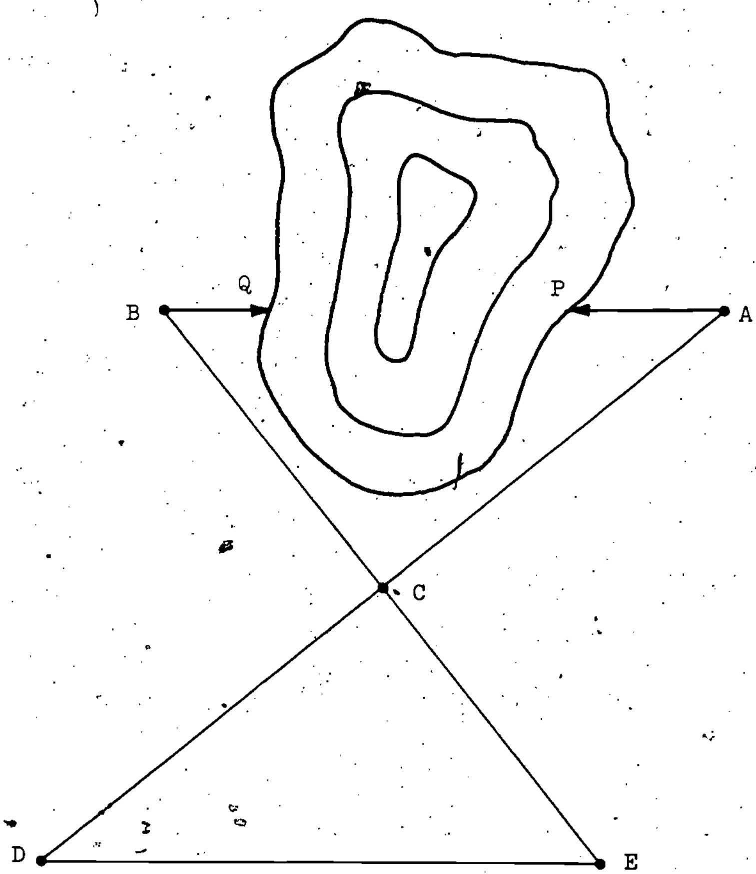


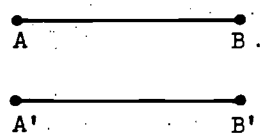
Figure 3

The ray $\overrightarrow{AP}$ was drawn so that $\angle A$ had the same measure as $\angle D$. Likewise the ray $\overrightarrow{BQ}$ was drawn so that the measures of $\angle B$ and $\angle E$ were equal.

Two construction teams started tunneling at P and Q on the same level in the directions indicated by the rays $\overrightarrow{AP}$ and $\overrightarrow{BQ}$. Sure enough, they met in the middle of the mountain.

In this chapter we shall discuss the geometric ideas which enabled the ancient Greeks to solve this problem.

8-2. Congruent Segments and Congruent Angles



On the edge of a ruler, the marks that match points A and B of $\overline{AB}$ maybe made to coincide with points A' (read A prime) and B' (read B prime). This shows that $\overline{A'B'}$ is a copy of $\overline{AB}$ and that they have the same length. In other words, they are congruent.

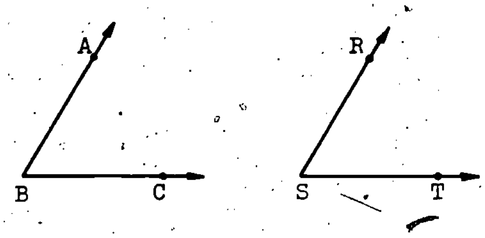
If we let "$\cong$" mean "is congruent to", then we can write:

$$\overline{A'B'} \cong \overline{AB}.$$

We use the names A' and B' to emphasize that these points correspond respectively to points A and B. Since $\overline{AB}$ and $\overline{A'B'}$ have the same length, we can write, "$m\overline{AB} = m\overline{A'B'}$", which means "the measure of $\overline{AB}$ equals the measure of $\overline{A'B'}$".

Definition: For any segments $\overline{AB}$ and $\overline{CD}$, if $m\overline{AB} = m\overline{CD}$, then $\overline{AB} \cong \overline{CD}$. Furthermore, if $\overline{AB} \cong \overline{CD}$ then $m\overline{AB} = m\overline{CD}$.

We also use the symbol AB for $m\overline{AB}$. Thus $\overline{AB}$ stands for the set of points which make up the segment, while AB stands for the real number which is the measure of the segment.



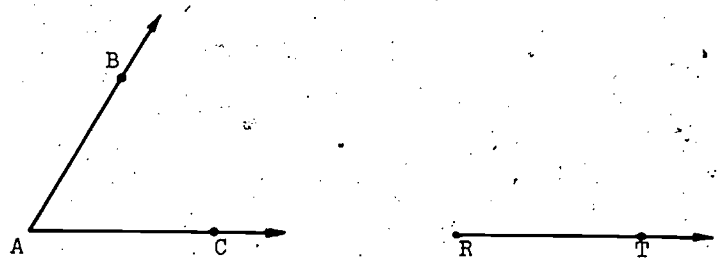
If a copy of $\angle ABC$ is made on a tracing sheet, it will fit exactly on or coincide with $\angle RST$. Then the two angles are congruent and they have the same measure.

Definition: For any angles $\angle ABC$ and $\angle RST$, if $m\angle ABC = m\angle RST$ then $\angle ABC \cong \angle RST$. Furthermore, if $\angle ABC \cong \angle RST$ then $m\angle ABC = m\angle RST$.

Exercises 8-2a

(Class Discussion)

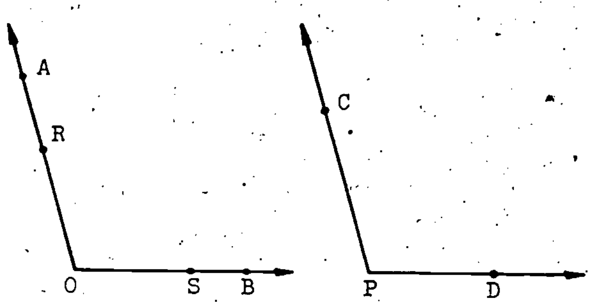
1. Copy the diagram below.



Show how a protractor can be used to draw a ray $\overrightarrow{RS}$ so that $\angle SRT \cong \angle BAC$.

2. Explain why any segment or any angle is always congruent to itself.

3. The two angles, $\angle AOB$ and $\angle CPD$ are congruent.



(a) Explain why $\angle AOB = \angle ROS$ but $\angle AOB \neq \angle CPD$.

(b) What is the difference between the meaning of "equals" $(=)$ and of "is congruent to" $(\cong)$?

(c) Why do we write: $m\angle AOB = m\angle CPD$, instead of: $m\angle AOB \cong m\angle CPD$?

4. Given $\overline{AB} \cong \overline{BC}$. Draw sketches and give reasons for your answers to the following.

(a) Can points A, B, and C be on a line?

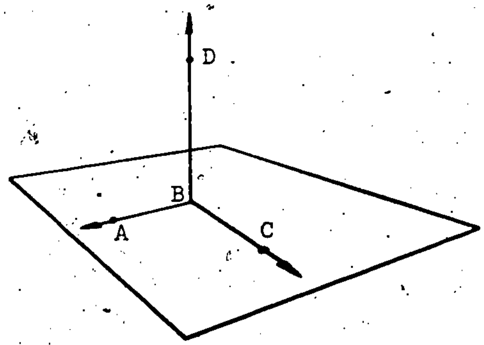
(b) Must points A, B, and C be on a line?

(c) Can points A, B, and C be in a plane?

(d) Must points A, B, and C be in a plane?

5. Given $\angle ABC \cong \angle CBD$.

(a) Can A, B, C, and D be in a plane? Show a sketch.



(b) Use the diagram at the left to show that points A, B, C, and D do not have to be in the same plane in order that $\angle ABC \cong \angle CBD$.

(c) Show that the corner of a room can also be used to show that A, B, C, and D need not be in the same plane.

Exercises 8-2b

1. Explain the difference between the following two statements: $\overline{AB} = \overline{CD}$; $\overline{AB} \cong \overline{CD}$.

2. Tell whether each of the following is a number, or a set of points.

(a) $\overline{CD}$ (c) $m\angle CAB$ (e) CD

(b) $m\overline{AB}$ (d) $\angle DCA$

3. Tell whether each of the following is true or false and give a reason for your answer.

(a) If $m\angle A = m\angle D$ then $\angle A = \angle D$.

(b) If $\angle M = \angle R$ then $m\angle M = m\angle R$.

(c) If $\angle B \cong \angle C$ then $\angle B = \angle C$.

(d) If $\angle P = \angle Q$ then $\angle P \cong \angle Q$.

For each of the following exercises, draw sketches and give reasons for your answers.

4. If $\overline{AB} \cong \overline{BC}$ and $\overline{BC} \cong \overline{CD}$,

(a) can A, B, C, and D be on a line?

(b) must A, B, C, and D be on a line?

5. If $\overline{AB} \cong \overline{BC}$ and $\overline{BC} \cong \overline{CA}$,

(a) can A, B, and C be on the same line?

(b) what is the figure called which contains these three congruent segments?

6. If $\angle AOB \cong \angle BOC \cong \angle COA$,

(a) can A, B, C, and O be in a plane?

(b) must A, B, C, and O be in a plane?

7. If $\overline{AB} \cong \overline{BC}$, $\overline{BC} \cong \overline{CD}$, and $\overline{CD} \cong \overline{DA}$,

(a) can A, B, C, and D be on a line?

(b) can A, B, C, and D be in a plane?

8. If $\angle DAB \cong \angle DCB$ and $\angle CDA \cong \angle ABC$, can A, B, C, and D be in a plane?

9. If $\overline{AB} \cong \overline{BC}$ and $\overline{CD} \cong \overline{DA}$, can A, B, C, and D be in a plane?

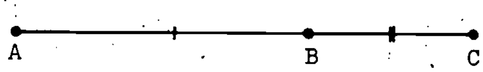
10. If $\overline{AB} \cong \overline{CD}$ and $\overline{AC} \cong \overline{BD}$,

(a) can A, B, C, and D be on a line?

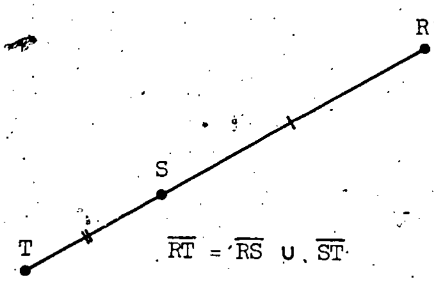
(b) must A, B, C, and D be on a line?

8-3. Addition and Subtraction Properties for Segments

In the drawing below, $\overline{AB}$ and $\overline{SR}$ have been marked with a single mark to show that $\overline{AB} \cong \overline{SR}$. Likewise $\overline{BC}$ and $\overline{ST}$ are marked with double marks to show that $\overline{BC} \cong \overline{ST}$.



$\overline{AC} = \overline{AB} \cup \overline{BC}$



$\overline{RT} = \overline{RS} \cup \overline{ST}$

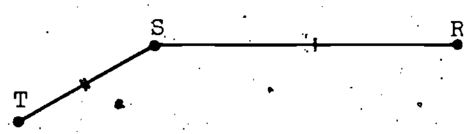
We know that $m\overline{AB} = m\overline{RS}$ and $m\overline{BC} = m\overline{ST}$. Why? It seems reasonable that the measures of $\overline{AC}$ and $\overline{RT}$ can be found by adding the measures of the two smaller segments contained in $\overline{AC}$ and $\overline{RT}$.

Then, $$m\overline{AC} = m\overline{AB} + m\overline{BC}$$

and $$m\overline{RT} = m\overline{RS} + m\overline{ST}.$$

The measures of $\overline{AC}$ and $\overline{RT}$ are found by adding the same pair of numbers. Why?

Hence $m\overline{AC} = m\overline{RT}$, and therefore $\overline{AC} \cong \overline{RT}$. We must make certain that the points are collinear because if $\overline{RS}$ and $\overline{ST}$ are drawn as shown at the left, then $\overline{RT}$ and $\overline{AC}$ are not congruent.

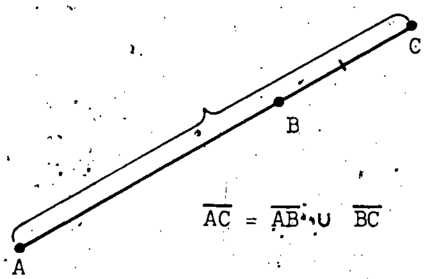


We shall agree to the following statement.

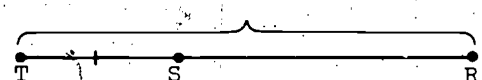
Addition Property for Congruence of Segments:

If B is between A and C, and S is between R and T, and if $\overline{AB} \cong \overline{RS}$ and $\overline{BC} \cong \overline{ST}$, then $\overline{AC} \cong \overline{RT}$.

This property may also be observed in subtraction form. In the drawing below, $\overline{AC} \cong \overline{RT}$ as shown by the braces and $\overline{BC} \cong \overline{ST}$ as shown by the single mark.



$\overline{AC} = \overline{AB} \cup \overline{BC}$



$\overline{RT} = \overline{RS} \cup \overline{ST}$

The measures of $\overline{AB}$ and $\overline{RS}$ may be found by subtracting the same pair of numbers.

$$m\overline{AB} = m\overline{AC} - m\overline{BC}$$

$$m\overline{RS} = m\overline{RT} - m\overline{ST}$$

Therefore $\overline{AB} \cong \overline{RS}$.

This property is called the <u>subtraction property of congruence for segments</u>.

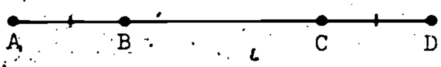
Exercises 8-3a

(Class Discussion)

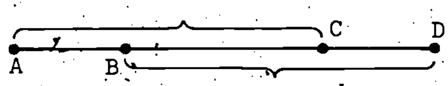
1. Illustrate the addition and subtraction properties for segments using sticks.

2. State the subtraction property of congruence for segments.

3. For the segment $\overline{AD}$ shown below, $\overline{AB} \cong \overline{CD}$. What conclusion can you make, using the addition property for segments?



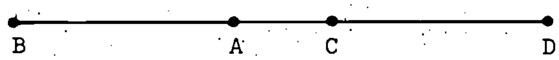
4. For the segment $\overline{AD}$ below, $\overline{AC} \cong \overline{BD}$ as shown by the braces. What conclusion can you make, using the subtraction property for segments?



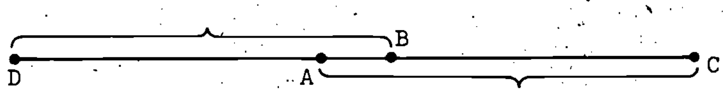
Exercises 8-3b

From the information given, state the conclusions you can make and justify your answers.

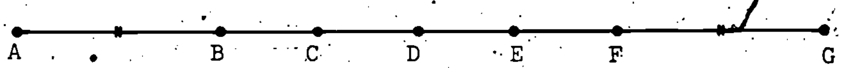
1. Given: $\overline{AB} \cong \overline{CD}$.



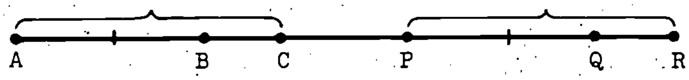
2. Given: $\overline{AC} \cong \overline{BD}$.



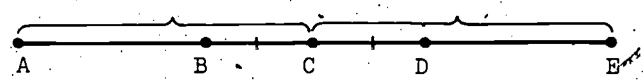
3. Given: $\overline{AB} \cong \overline{FG}$.



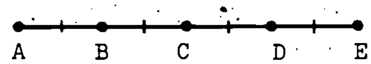
4. Given: $\overline{AC} \cong \overline{PR}$, $\overline{AB} \cong \overline{PQ}$.



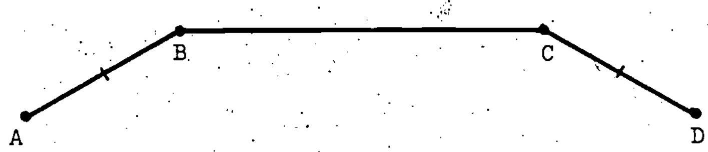
5. Given: $\overline{AC} \cong \overline{CE}$, $\overline{BC} \cong \overline{CD}$.



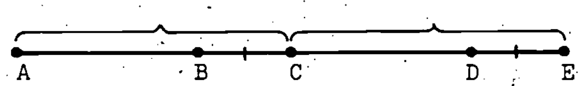
6. Given: $\overline{AB} \cong \overline{BC}$, $\overline{BC} \cong \overline{CD}$, $\overline{CD} \cong \overline{DE}$.



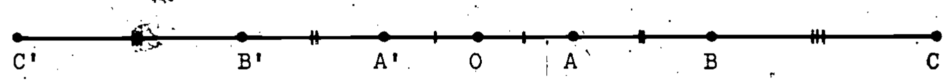
7. Given: $\overline{AB} \cong \overline{CD}$.



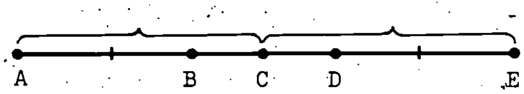
8. Given: $\overline{AC} \cong \overline{CE}$, $\overline{BC} \cong \overline{DE}$.



9. Given: $\overline{OA} \cong \overline{OA'}$, $\overline{AB} \cong \overline{A'B'}$, $\overline{BC} \cong \overline{B'C'}$.

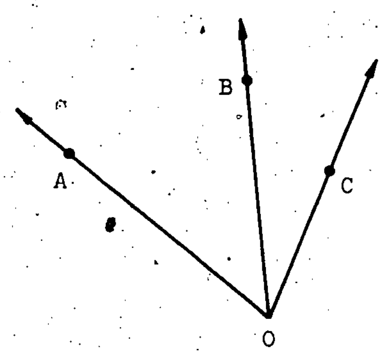


10. Given: $\overline{AC} \cong \overline{CE}$, $\overline{AB} \cong \overline{DE}$.



8-4. Addition and Subtraction Properties for Angles

In the diagram below, the ray $\overrightarrow{OB}$ is between $\overrightarrow{OA}$ and $\overrightarrow{OC}$ since B is in the interior of $\angle AOC$.



Thus, two angles are formed; $\angle AOB$ and $\angle BOC$, which have a common vertex, a common ray, and their interiors do not overlap.

These two angles are called adjacent angles. (Adjacent means "neighboring".)

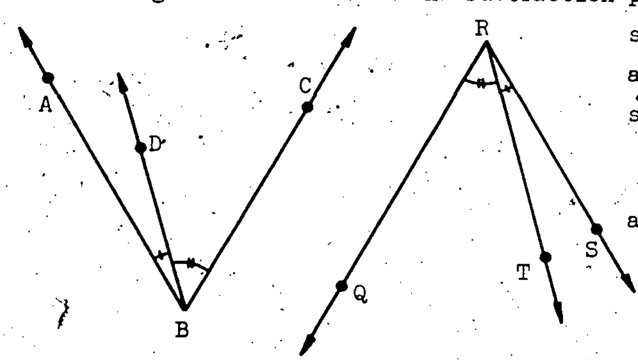
Definition: When we say that two angles are adjacent, we mean that (a) they have a common vertex, (b) they have a common ray or side, and (c) their interiors do not overlap.

Angles have addition and subtraction properties similar to those for segments. For the adjacent angles shown at the left, the single marks show that

$$\angle ABD \cong \angle TRS$$

and the double marks show that

$$\angle DBC \cong \angle QRT.$$



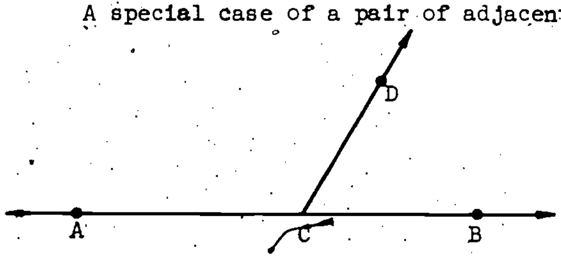
The measures of $\angle ABC$ and $\angle QRS$ are found by adding the same pair of numbers.

$$m\angle ABC = m\angle ABD + m\angle CBD$$

$$m\angle QRS = m\angle SRT + m\angle QRT$$

Therefore, $\angle ABC \cong \angle QRS$.

A special case of a pair of adjacent angles is shown below. On a line $\overleftrightarrow{AB}$, C is between A and B forming opposite rays $\overrightarrow{CA}$ and $\overrightarrow{CB}$. A point D not on the line determines the ray $\overrightarrow{CD}$, forming the two adjacent angles $\angle ACD$ and $\angle BCD$.

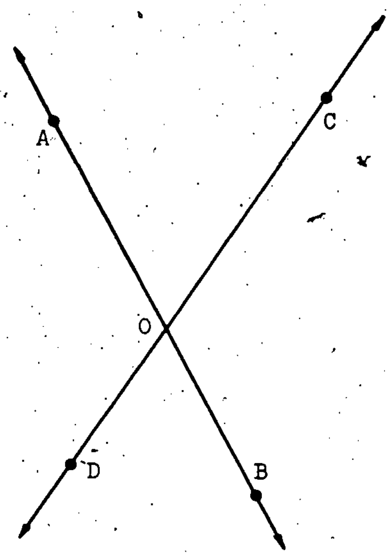


$m\angle ACD + m\angle BCD = 180$

Using a protractor it can be shown that the sum of the measures of these two angles, in degrees, is 180. The measure of either angle can be expressed in subtraction form.

For example, $m\angle BCD = 180 - m\angle ACD$.

Two lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at O as shown below.



Name the four angles that are formed.

$\angle AOC$ and $\angle BOD$ are a pair of <u>non-adjacent</u> angles. Why?

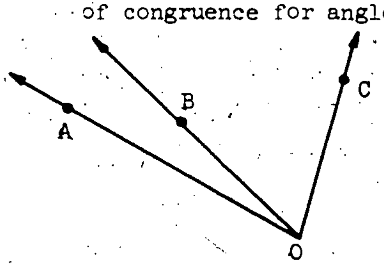
Name the other pair of non-adjacent angles.

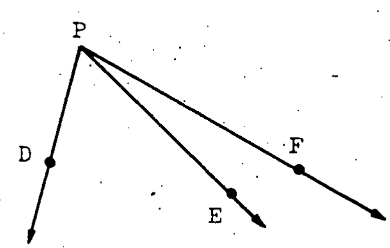
These two pairs of non-adjacent angles formed by two intersecting lines are called <u>vertical angles</u>. (Note that here "vertical" is not associated with "horizontal".)

<u>Exercises 8-4a</u>

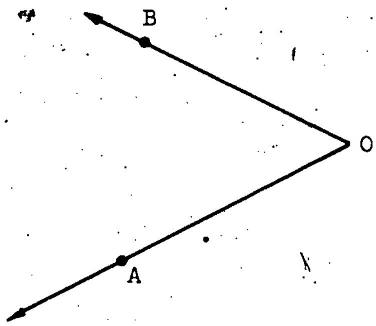
(Class Discussion)

1. Use the diagram below to state the addition and subtraction properties of congruence for angles.





2. For a given angle, $\angle AOB$, describe how you would draw



(a) an angle $\angle AOD$ that is adjacent to $\angle AOB$ so that the sum of the measures of the two angles is 180.

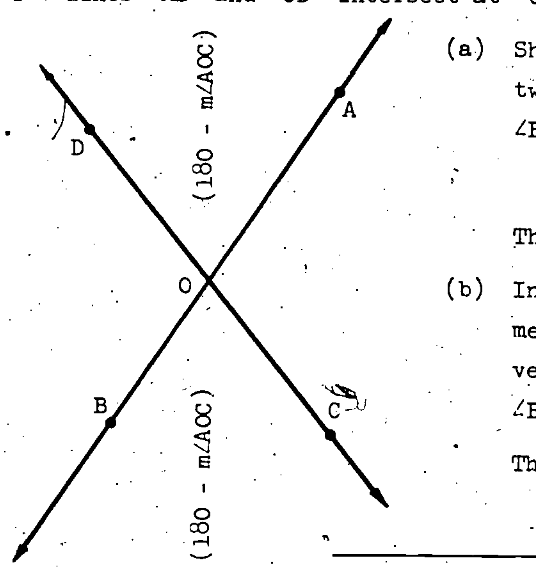
(b) an angle $\angle BOD$ that is not adjacent to $\angle AOB$.

(c) an angle $\angle CQD$ so that $\angle AOB$ and $\angle COD$ are vertical angles.

3. Describe the figure formed by the union of any pair of vertical angles.

4. Draw at least three pairs of vertical angles. Use a protractor and find the measures of these angles. What do you suspect is true about the measures of each pair of vertical angles?

5. Two lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at O as shown.



(a) Show that the measures of the two vertical angles $\angle AOD$ and $\angle BOC$ are equal to

$180 - m\angle AOC$.

Therefore, $\angle AOD \cong \angle BOC$. Why?

(b) In the same way, show that the measures of the other pair of vertical angles, $\angle AOC$ and $\angle BOD$, are equal.

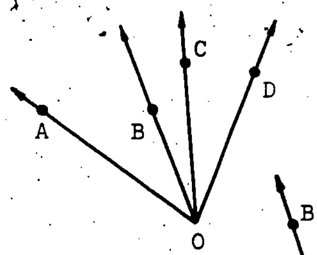
Therefore, $\angle AOC \cong \angle BOD$. Why?

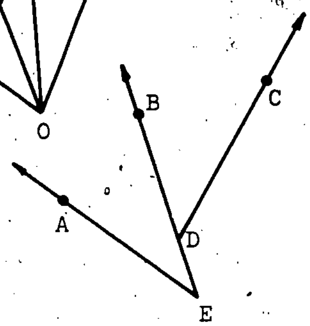
From Exercise 5 above, we may conclude that:

Any pair of vertical angles are congruent.

Exercises 8-4b

1. From the diagrams shown below, explain why the following pairs of angles are not adjacent.



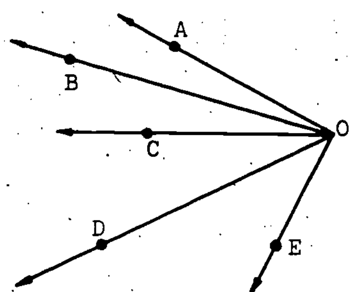


(a) ∠AOB and ∠COD

(b) ∠AOB and ∠AOC

(c) ∠AOC and ∠BOD

(d) ∠AEB and ∠BDC

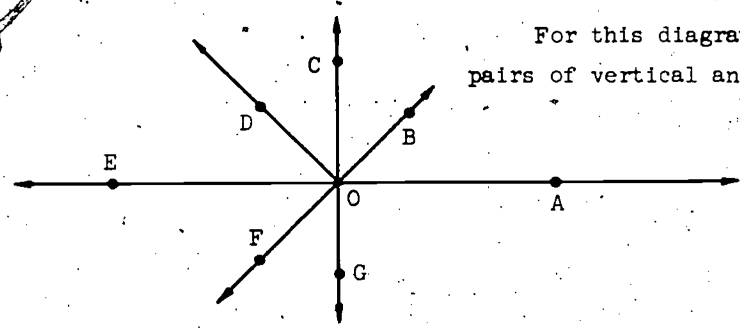
2. 

For this diagram, name all the angles that are adjacent to

(a) ∠AOB.

(b) ∠COD.

(c) ∠BOD.

3. 

For this diagram, name all the pairs of vertical angles.

4. For a given plane, how many pairs of vertical angles are formed by

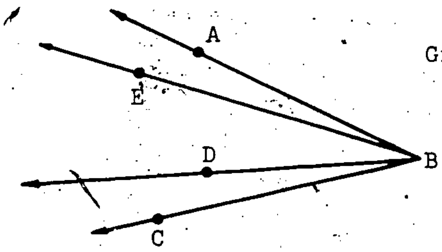
(a) two intersecting lines?

(b) three lines intersecting at a point?

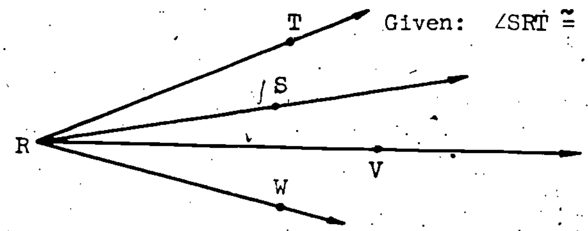
(c) four lines intersecting at a point?

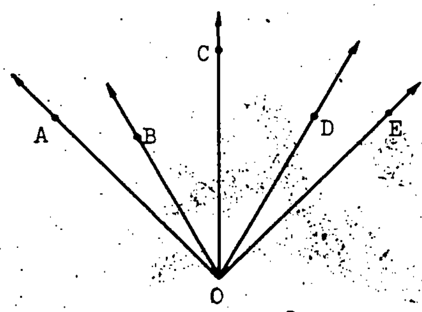
In each exercise below, state the conclusions you can make from the given information, and justify your answer.

5. Given: $\angle ABD \cong \angle CBE$



6. Given: $\angle SRT \cong \angle VRW$





Exercises 7, 8, and 9

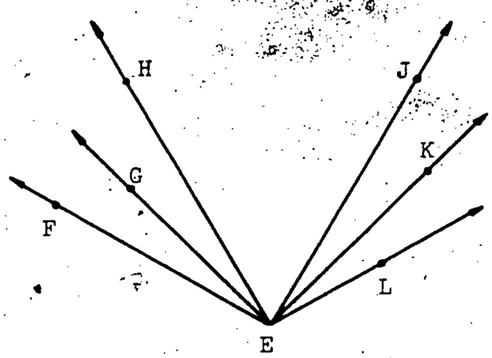
7. Given: $\angle AOB \cong \angle EOD$

$\angle BOC \cong \angle COD$

8. Given: $\angle AOC \cong \angle EOC$

$\angle AOB \cong \angle EOD$

9. Given: $\angle AOD \cong \angle BOE$



Exercises 10, 11, 12, 13, 14

10. Given: $\angle FEG \cong \angle LEK$

11. Given: $\angle FEH \cong \angle LEJ$

12. Given: $\angle FEK \cong \angle GEL$

13. Given: $\angle FEG \cong \angle LEK$

$\angle GEH \cong \angle JEK$

14. Given: $\angle KEH \cong \angle GEJ$

8-5. Copying Triangles

In the problem of the ancient Greeks building a tunnel, we saw an application of making a copy of a triangle. In this section we shall study the problem of copying triangles and see just how much information about a triangle must be known in order to make a copy of it.

We could make a copy of a triangle $\triangle ABC$ by making a cardboard cut-out which just fits the triangle, then moving the cut-out to a new position and carefully tracing around it. This results in the $\triangle A'B'C'$ as shown below where $\triangle ABC \cong \triangle A'B'C'$.

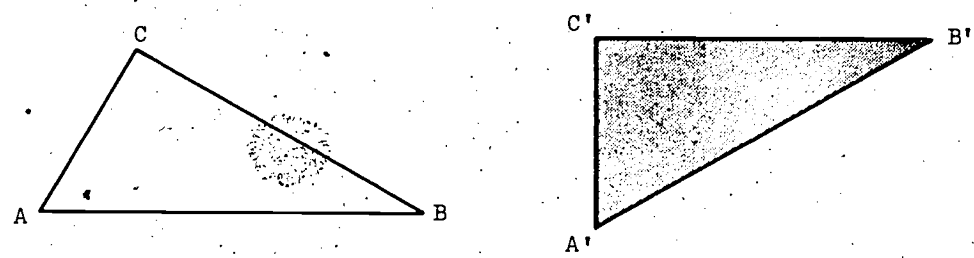


Figure 4

We shall consider, however, the problem of copying one such triangle without making the cardboard cut-out.

Let us start with the same triangle $\triangle ABC$ and a segment $\overline{A'B'}$ where $\overline{A'B'} \cong \overline{AB}$.

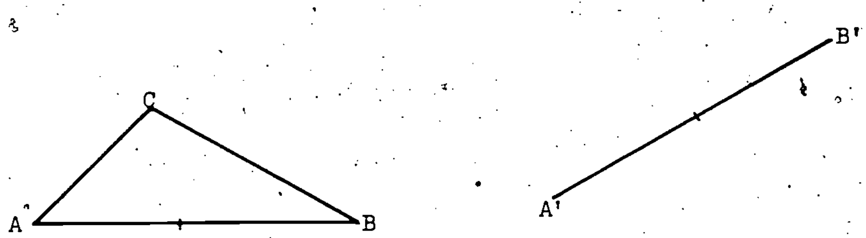


Figure 5

We wish to construct the triangle $\triangle A'B'C'$ which was obtained by the cardboard cut-out shown in Figure 4. The problem consists simply of locating point C' because when the location of C' is found the triangle $\triangle A'B'C'$ can be drawn.

Three methods for locating C' will be shown. These methods involve the ability to measure some, but not all, of the sides and angles of $\triangle ABC$.

First Method

The first method for locating C' consists of using a protractor and drawing a ray from A' so that $m\angle A' = m\angle A$. Point C' must be on this ray.

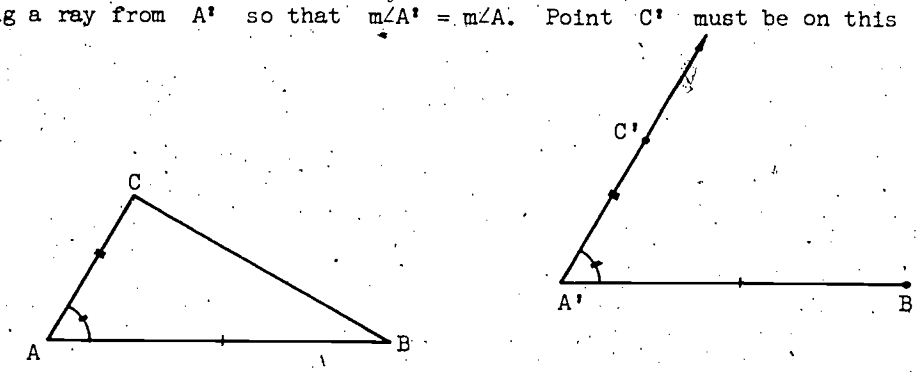


Figure 6

We locate C' on this ray so that $m\overline{A'C'} = m\overline{AC}$. Then $\triangle A'B'C'$ is completed by drawing $\overline{B'C'}$.

To copy a triangle, then, it is only necessary to be able to measure two sides and the included angle. $\angle CAB$ is said to be included between the sides $\overline{AC}$ and $\overline{AB}$ because these sides are contained in the angle.

SAS (side, angle, side) Congruence Property:

If two sides and the included angle of one triangle have the same measures as two sides and the included angle of another triangle, then the two triangles are congruent.

Second Method

Start again with $\triangle ABC$ and $\overline{A'B'}$. Draw from A' a ray so that $m\angle A' = m\angle A$. Then draw from B' a ray so that $m\angle B' = m\angle B$.

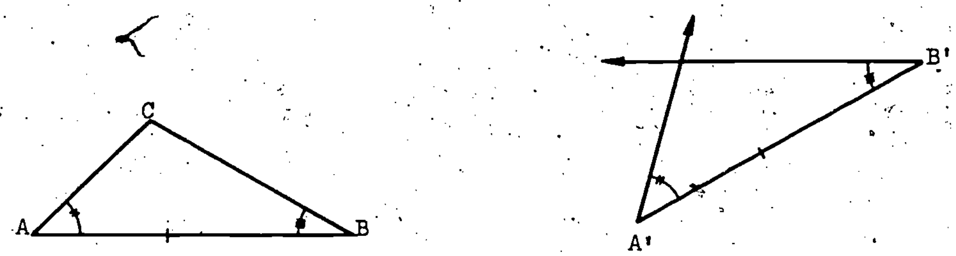


Figure 7

Point C' must lie on both rays and therefore is the point of intersection of the two rays.

In this case, to copy a triangle it is only necessary to measure two angles and the included side. $\overline{AB}$ is said to be included between $\angle A$ and $\angle B$ because these angles contain the side.

ASA (angle, side, angle) Congruence Property:

> If two angles and the included side of one triangle have the same measures as two angles and the included side of another triangle, then the triangles are congruent.

Third Method

This time we shall measure only the sides of $\triangle ABC$ and none of the angles.

In Figure 8 below, point C' must lie on the circle with center at A' and radius equal to $m\overline{AC}$.

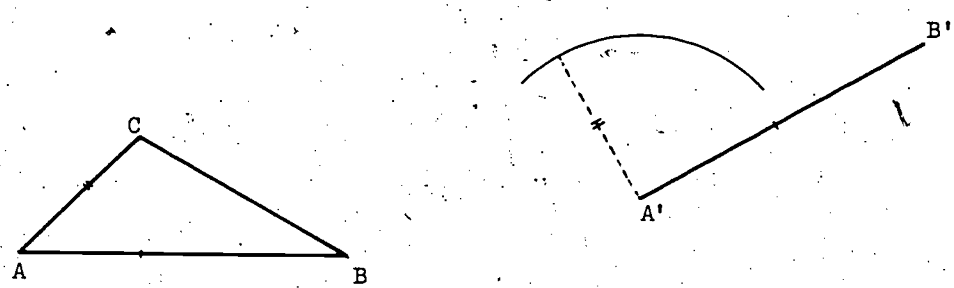


Figure 8

Similarly, point C' must lie on the circle with center at B' and radius equal to BC.

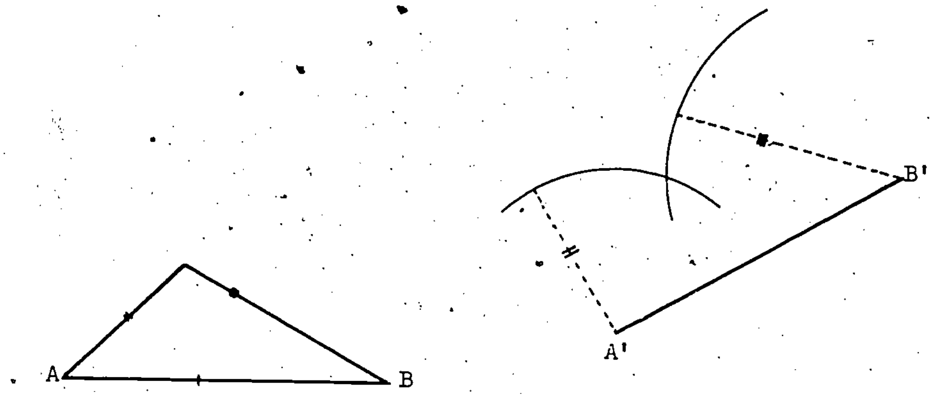


Figure 9

Since C' lies on both these circles above the segment $\overline{A'B'}$, then it must be the point of intersection of the arcs shown in Figure 9.

Using this method to copy a triangle, it is only necessary to be able to measure the three sides.

SSS Congruence Property:

> If the three sides of one triangle have the same measures as the three sides of another triangle, then the triangles are congruent.

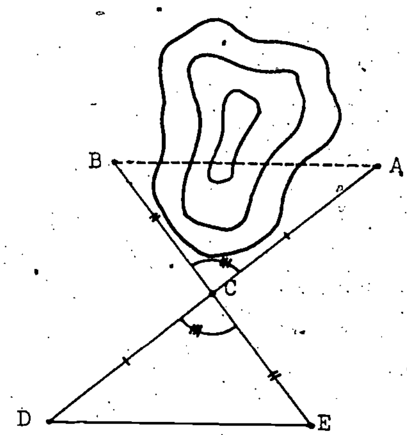


Figure 10

Returning to the tunneling problem of the ancient Greeks, we see that the two triangles, $\triangle ABC$ and $\triangle DEC$, are congruent by the SAS property. Therefore $\angle A \cong \angle D$ and $\angle B \cong \angle E$ which determine the directions of the tunneling from both sides of the mountain.

Exercises 8-5

1. Perform the following constructions using a ruler and a protractor. (Lengths are expressed in inches.)

   (a) Construct $\triangle ABC$ so that $BC = 1\frac{1}{2}$, $m\angle CBA = 75$, and $AB = 1$.

   (b) Construct $\triangle DEF$ so that $DE = 1\frac{1}{2}$, $EF = 1\frac{3}{4}$, and $DF = 2$.

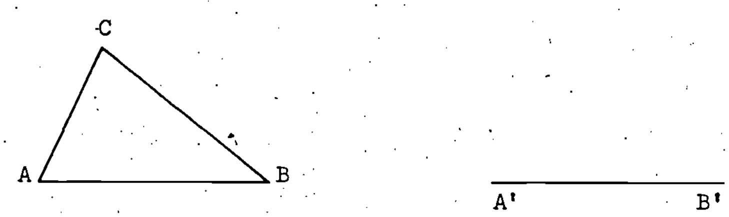
   (c) Construct $\triangle GHJ$ so that $m\angle GHJ = 50$, $HJ = 1\frac{1}{2}$, and $m\angle GJH = 60$.

   (d) Construct $\triangle KLR$ so that $LR = 1\frac{3}{4}$, $m\angle KLR = 70$, and $KL = 2$.

2. For each of the constructions in problem 1 find the measures of the sides and angles whose measures were not given.

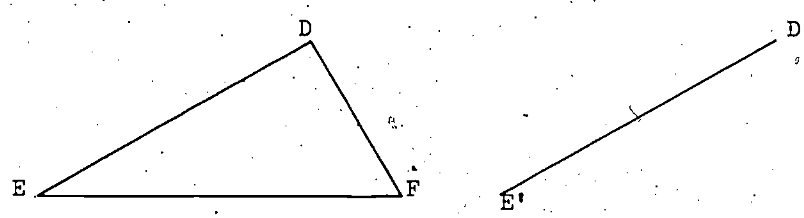
3. Copy the given triangle and segment. Then find the missing vertex using the congruence property indicated.

   (a)



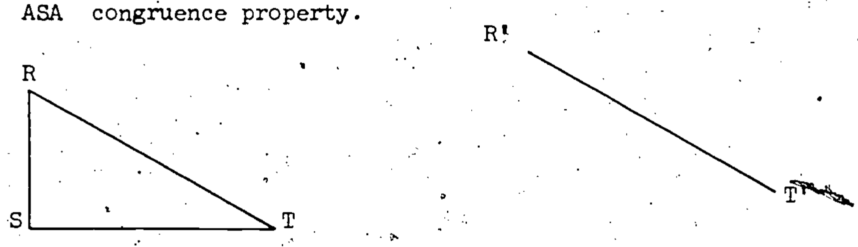
   Use the SAS congruence property.

   (b)



   Use the ASA congruence property.

   (c)

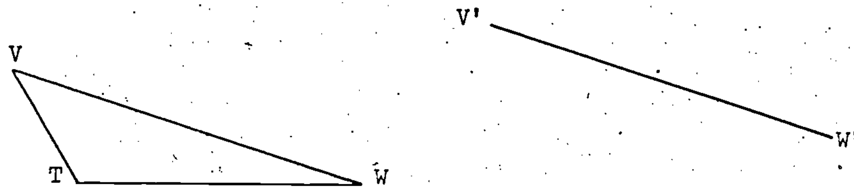


   Use the SSS congruence property.

(d)

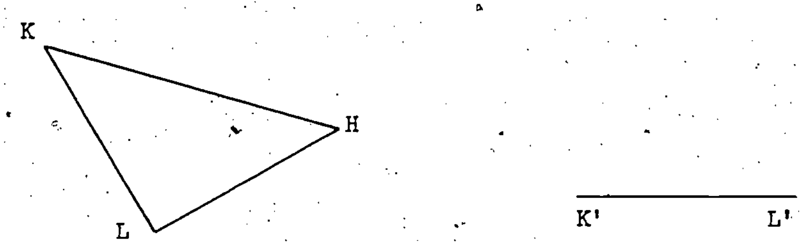
Use the ASA congruence property.

(e)



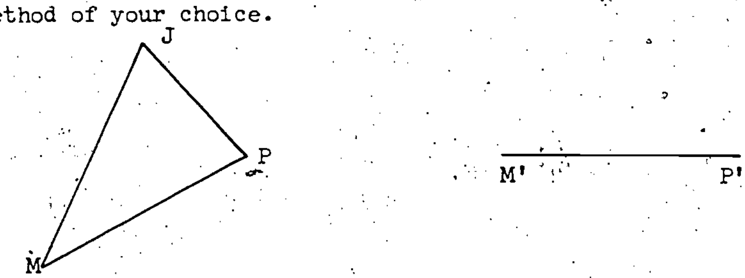
Use the SSS congruence property.

(f)



Use a method of your choice.

(g)



Use a method of your choice.

8-6. Congruent Triangles and Correspondence

We have been using A', B', C' as names for the vertices of a triangle that is congruent to $\triangle ABC$ in a specially convenient way. If a copy of $\triangle ABC$ were fitted on $\triangle A'B'C'$ then the vertices A and A', B and B', C and C' would match or correspond. They are called corresponding vertices.

The corresponding vertices of two congruent triangles are named by the order they are written in a congruence statement as shown below.

$\triangle ABC \cong \triangle KJL$

<u>Corresponding Vertices</u>

$A \longleftrightarrow K$

$B \longleftrightarrow J$

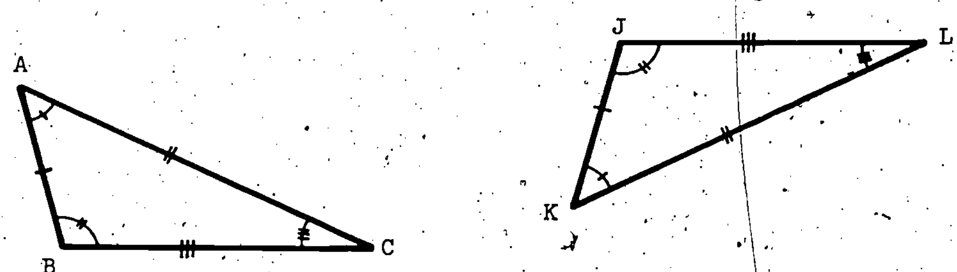
$C \longleftrightarrow L$

From the corresponding vertices, we can determine the corresponding sides and angles as follows:

$$\triangle ABC \cong \triangle KJL.$$

<u>Corresponding Sides</u>	<u>Corresponding Angles</u>
$\overline{AB} \cong \overline{KJ}$ or $AB = KJ$	$\angle A \cong \angle K$ or $m\angle A = m\angle K$
$\overline{AC} \cong \overline{KL}$ or $AC = KL$	$\angle B \cong \angle J$ or $m\angle B = m\angle J$
$\overline{BC} \cong \overline{JL}$ or $BC = JL$	$\angle C \cong \angle L$ or $m\angle C = m\angle L$

Thus a congruence statement gives us six facts at once. It is helpful to show the six corresponding congruent parts by the markings as shown below.



Exercises 8-6a

1. Explain why the following congruence statements show the same correspondence.

$$\triangle ABC \cong \triangle DEF$$

$$\triangle BAC \cong \triangle EDF$$

Write another congruence statement that shows the same correspondence.

2. Tell whether each of the following statements is true or false. Give a reason for each answer.

(a) A segment is congruent to itself.

(b) All the sides of a rectangle are congruent to each other.

(c) All the angles of a rectangle are congruent to each other.

(d) All the sides of a square are congruent to each other.

(e) All the sides of a quadrilateral are congruent to each other.

(f) A triangle is congruent to itself.

(g) Some triangles are congruent to squares.

3. If $\triangle RST \cong \triangle VWX$, complete the following:

$\overline{RT} \cong$ ________

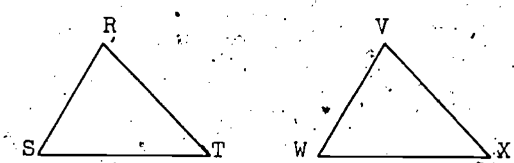
$\angle S \cong$ ________

$\overline{TS} \cong$ ________

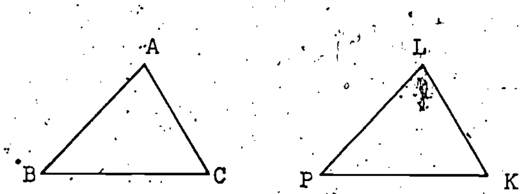
$\overline{RS} \cong$ ________

$\angle T \cong$ ________

$\angle R \cong$ ________



4. If $\triangle BCA \cong \triangle PKL$, list the six pairs of corresponding congruent parts.

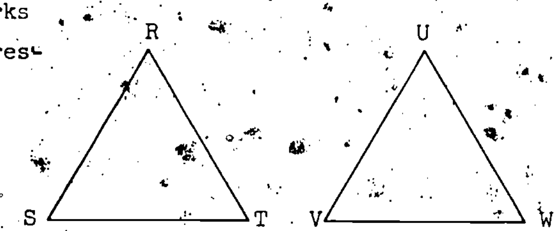


5. Given that $\triangle EFG \cong \triangle PRS$

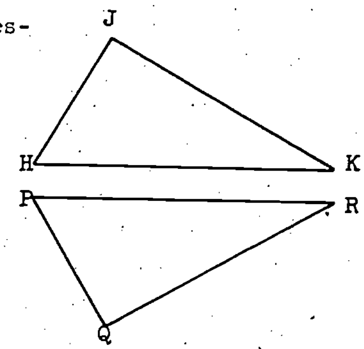
(a) Make a sketch of the congruent triangles.

(b) List the six pairs of corresponding congruent parts.

6. Given that $\triangle TRS \cong \triangle WUV$. Use marks to indicate the six pairs of corresponding congruent parts.



7. Given that $\triangle JHK \cong \triangle QPR$. Use marks to indicate the six pairs of corresponding congruent parts.



8. In each case, write a congruence for the two triangles that is determined by the given pairs of congruent parts:

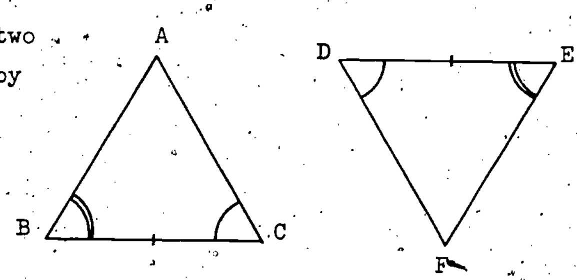
(a) $\overline{AD} \cong \overline{FG}$
$\overline{DE} \cong \overline{GH}$
$\overline{AE} \cong \overline{FH}$

(b) $\overline{PQ} \cong \overline{SR}$
$\angle Q \cong \angle R$
$\overline{QT} \cong \overline{RV}$

(c) $\overline{KL} \cong \overline{BC}$
$\angle KLM \cong \angle BCD$
$\overline{LM} \cong \overline{CD}$

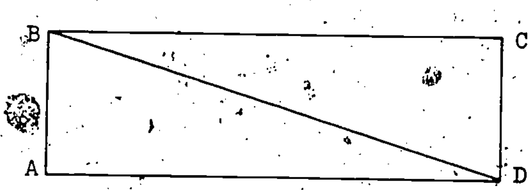
(d) $\angle ABC \cong \angle XYZ$
$\overline{AB} \cong \overline{XY}$
$\angle BAC \cong \angle YXZ$

9. Write the congruence for the two triangles that is determined by the marked pairs of congruent parts.

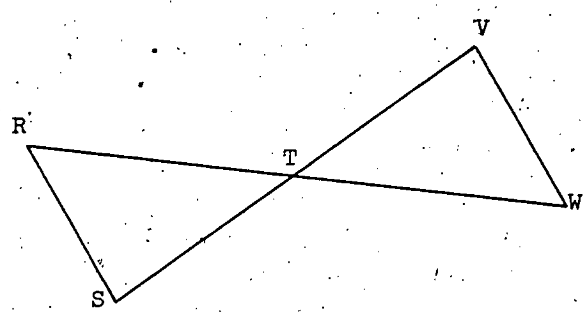


10. In the figures below, if a correspondence looks like a congruence assume that it really is a congruence. Write a congruence and then list the six pairs of corresponding congruent parts. Can you find more than one congruence for part (c)? More than one for part (d)?

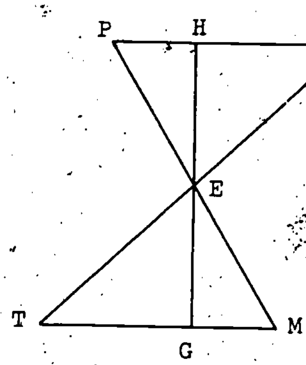
(a)



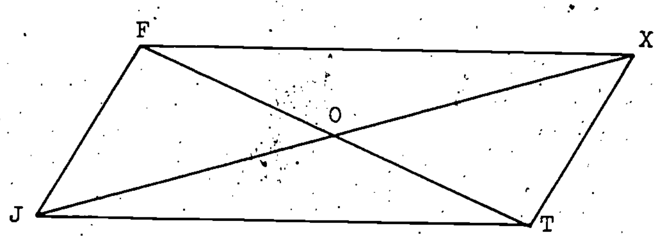
(b)



(c)



(d)



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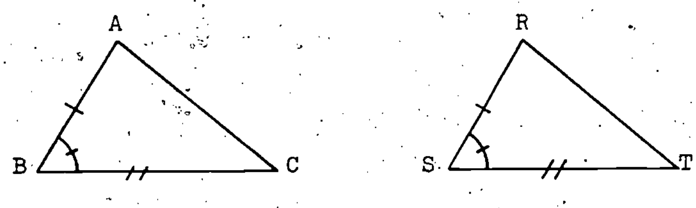
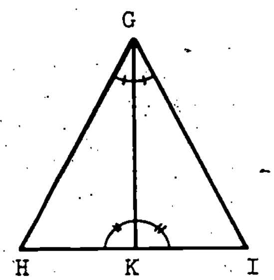
Exercises 8-6b

1. Each pair of triangles has markings indicating congruent parts. If the triangles are congruent, state the congruence and identify the congruence property used. If the given information does not enable you to reach the conclusion that the triangles are congruent, explain why.

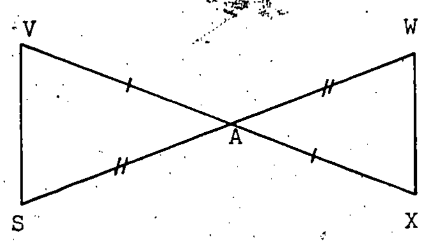
(a)



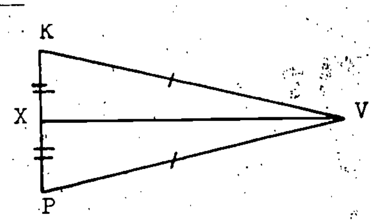
(b)



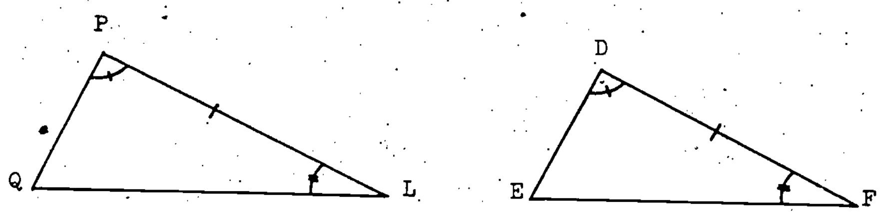
(c)



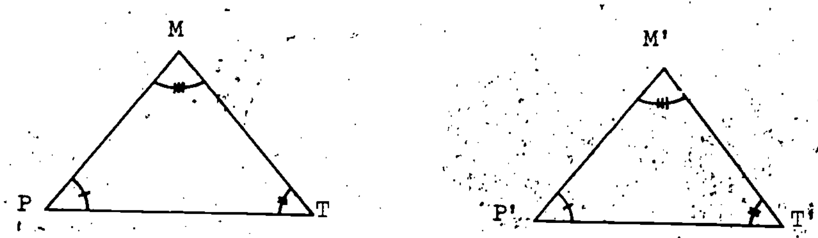
(d)



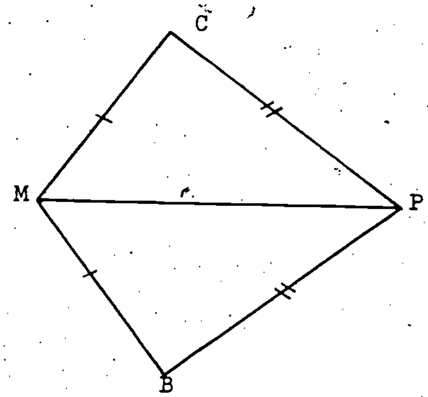
(e)



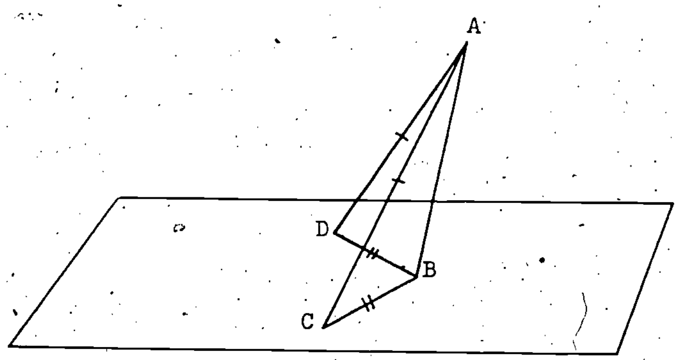
(f)



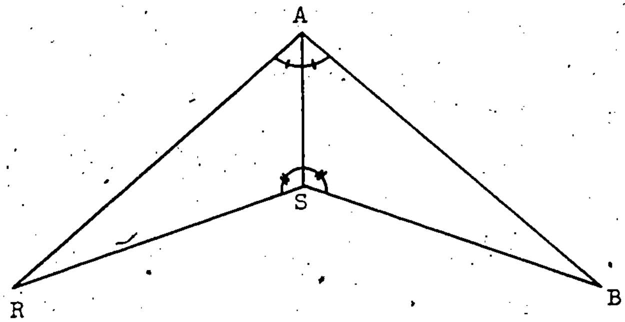
(g)



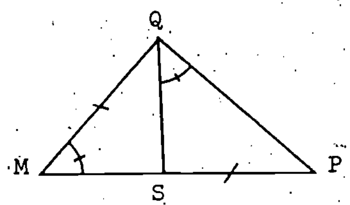
(h)



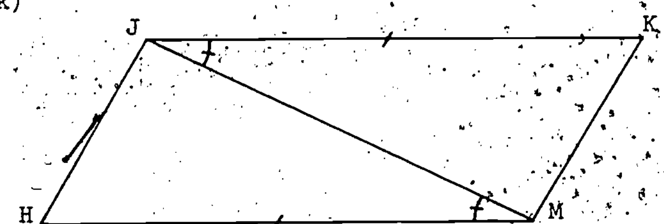
(i)



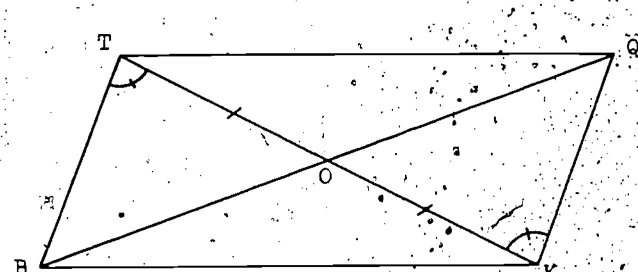
(j)



(k)



(ℓ)



61

2. It is given that

$\overline{AC} \cong \overline{VW}$

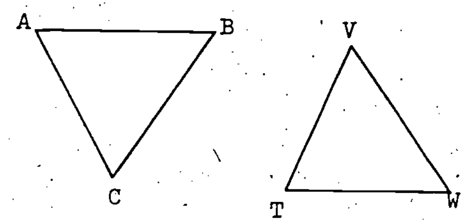
$\angle BAC \cong \angle VWT$

$\overline{AB} \cong \overline{TW}$

(a) Are the triangles congruent? Why?

(b) If so, write the congruence for the triangles.

(c) List the three pairs of corresponding congruent angles.



3. It is given that

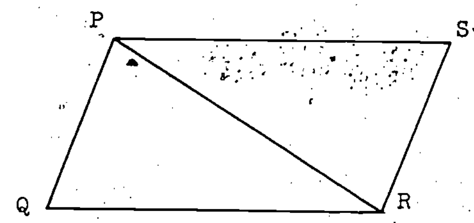
$\overline{PQ} \cong \overline{SR}$

$\overline{QR} \cong \overline{PS}$

(a) Are the triangles congruent? Why?

(b) If so, write the congruence for the triangles.

(c) List the additional pairs of corresponding congruent parts.



4. It is given that

$\angle YWX \cong \angle GFE$

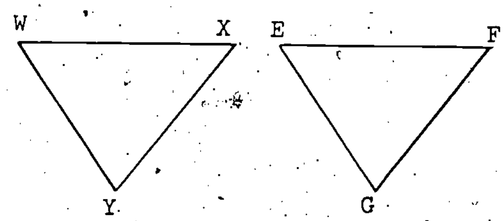
$\angle WXY \cong \angle FEG$

$\overline{WX} \cong \overline{EF}$

(a) Are the triangles congruent? Why?

(b) If so, write the congruence for the triangles.

(c) List the additional pairs of corresponding congruent parts.



5. It is given that

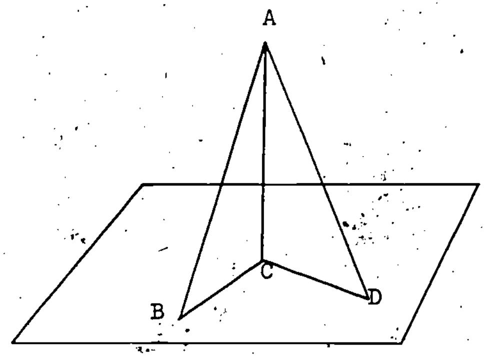
$\angle ACB \cong \angle ACD$

$\overline{BC} \cong \overline{CD}$

(a) Are the triangles congruent? Why?

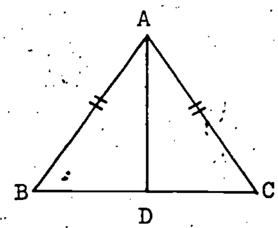
(b) If so, write the congruence for the triangles.

(c) List the additional pairs of corresponding congruent parts.

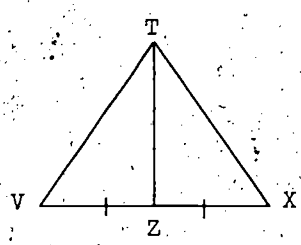


6. In each figure below, name an additional pair of congruent parts, such that if you know that these parts are congruent, you could say that the triangles are congruent. Write the congruence and identify the congruence property used.

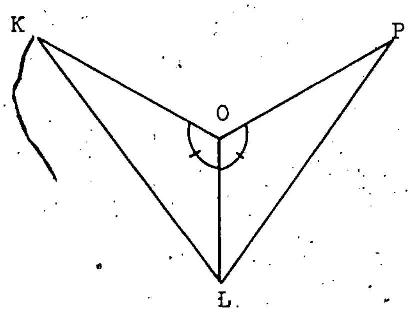
(a)



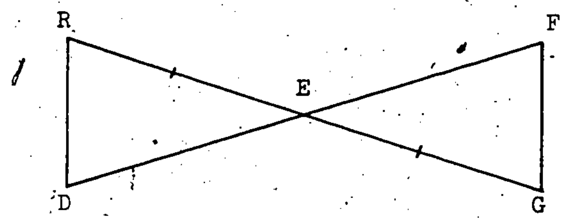
(b)



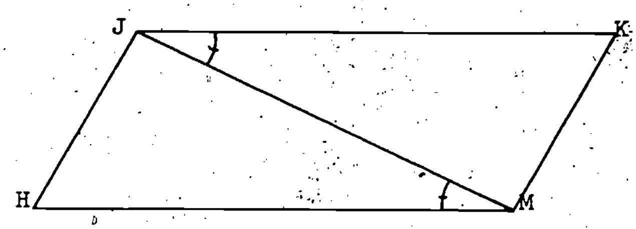
(c)



(d)



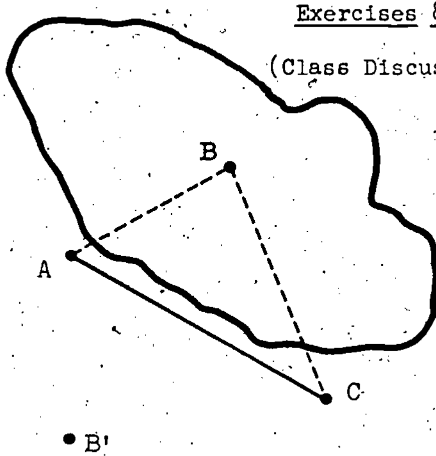
(e)



8-7. Some Applications of Congruence

Exercises 8-7a

(Class Discussion)



1. The problem is to measure the distance from point A on the shore to point B in the middle of a lake.

   Select a point C on the shore as shown.

   It is possible to measure $\angle A$ and $\angle C$.

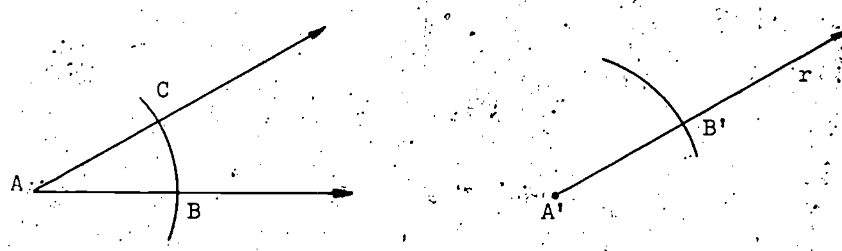
Show how you would find point B' so that AB = AB'.

What congruence property shows that $\triangle ABC \cong \triangle AB'C$?

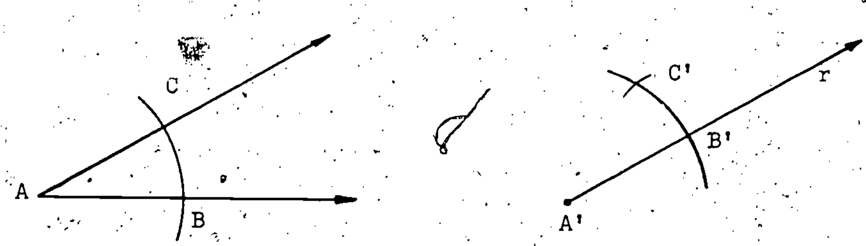
64

2. The problem is to make a copy of $\angle A$ with the vertex at A' and a side ray r as shown below without using a protractor.

Open your compass to an arbitrary amount and draw an arc with center at A intersecting the sides at B and C. Use the same compass setting and draw an arc with center at A' intersecting ray r at B' as shown below. C' must lie on this arc. Why?



Now adjust the compass so that the tips coincide with B and C. With this setting, draw an arc with center at B'. The intersection of the two arcs is at C'.



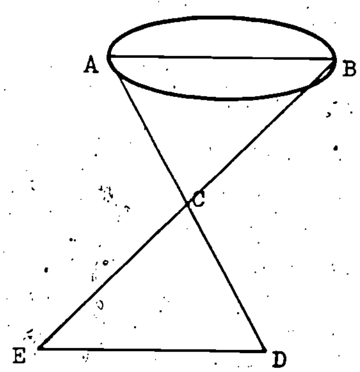
Complete the construction by drawing the ray $\overrightarrow{A'C'}$. What congruence property shows that $\triangle ABC \cong \triangle A'B'C'$ and therefore $\angle BAC \cong \angle B'A'C'$?

A triangle such as $\triangle ABC$ which has at least two congruent sides ($\overline{AB} \cong \overline{AC}$) is called an <u>isosceles triangle</u>.

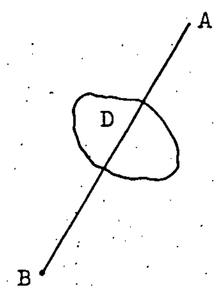
A triangle whose three sides are congruent is called an <u>equilateral triangle</u>.

<u>Exercises 8-7b</u>

1. If AB is the distance across a lake, show how $\overline{AB}$ may be measured if points A and B are accessible. Use the diagram at the right.



2. It is desired to measure the length of $\overline{AB}$ but a deep ditch at D intervenes. We can use congruent triangles to measure $\overline{AB}$. Explain how this may be done by each of the following methods.



A convenient point C is taken and $\overline{AC}$ and $\overline{BC}$ are drawn.

(a) Use Figure (a) and the ASA congruence property.

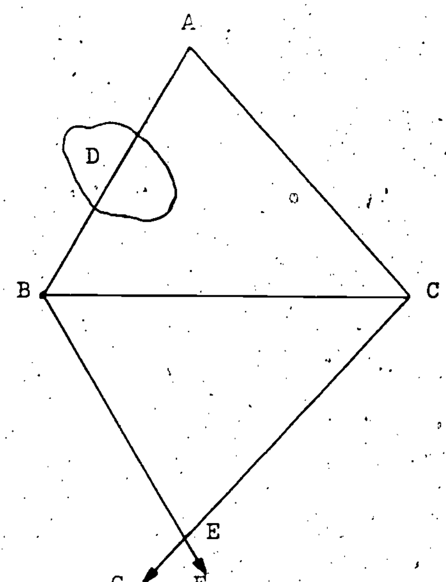


Figure (a)

(b) Use Figure (b) and the SAS congruence property.

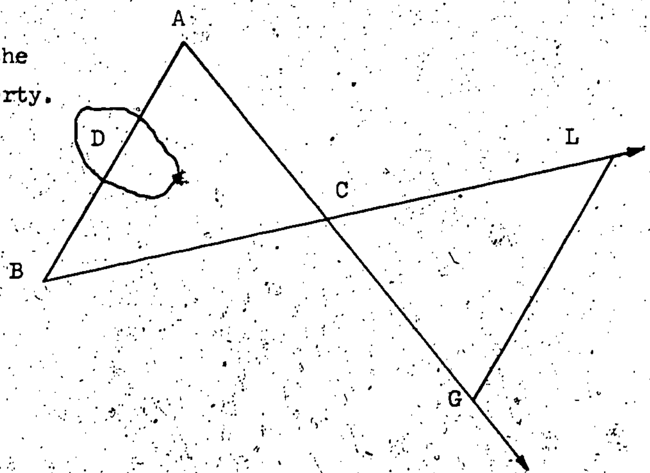
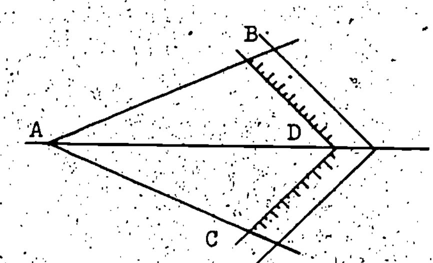
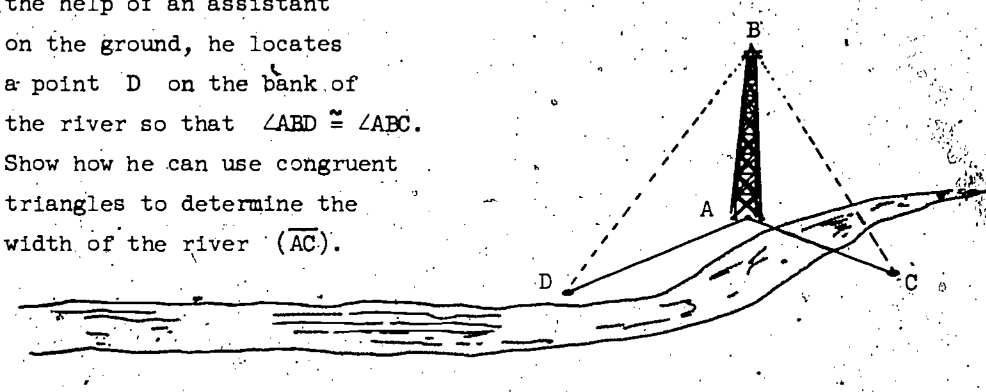


Figure (b)

3. In the figure at the right a carpenter's square is placed so that $\overline{AB} \cong \overline{AC}$ and $\overline{BD} \cong \overline{DC}$. Use congruent triangles to explain why $\angle BAD \cong \angle CAD$.



4. A surveyor at the top of a tower $(\overline{AB})$ sights across a river to point C. With the help of an assistant on the ground, he locates a point D on the bank of the river so that $\angle ABD \cong \angle ABC$. Show how he can use congruent triangles to determine the width of the river $(\overline{AC})$.



67

8-8. Congruent Figures and Motions

You have learned that two geometric figures are congruent to each other if they have exactly the same size and shape. Two figures in the same plane that are congruent may be made to fit together by a motion. In this part of the chapter we will carefully analyze some important types of motion. For example, the figure A on the left may be made to fit, or coincide with, the figure B by a sliding motion in the direction of the

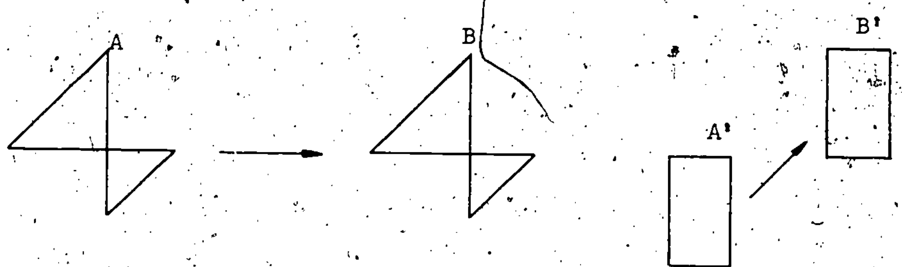


Figure 11

arrow. We say that figure B is the slide image of figure A, or that the slide maps figure A onto figure B. Likewise, figure B' is a slide image of figure A' because a sliding motion in the direction of the right-hand arrow carries A' into B'.

Consider the congruent figures shown below. Note that no slide will map figure R onto figure S.

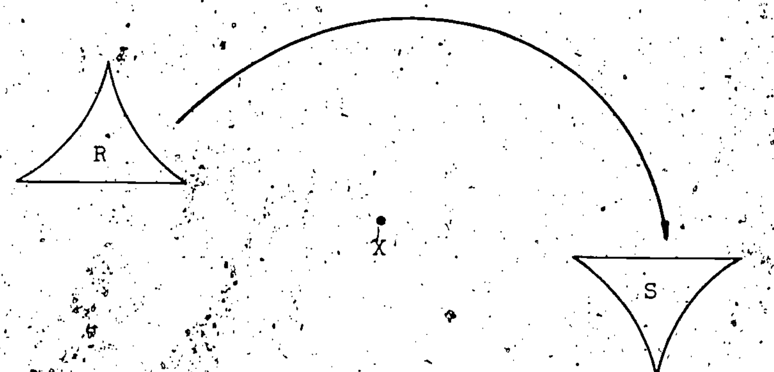


Figure 12

However, figure R may be made to coincide with figure S by a rotation or turn about point X as a center of turn. We say that figure S is the turn image of figure R, or that the turn maps figure R onto figure S.

The center of turn may even be a point common to the two figures. Consider the congruent figures shown below.

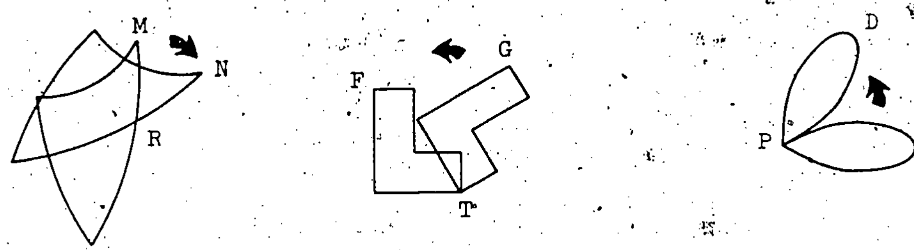


Figure 13

For each of these pairs of congruent figures, make two statements--one that uses the words <u>turn image</u> and one that uses the word <u>maps</u>. Also, name the point of turn in each case.

The third basic motion that we will consider is the flip. You can think of a flip as the motion involved when you turn over a sheet of paper. In the process of flipping a sheet of paper the plane of the paper is rotated, as shown below.

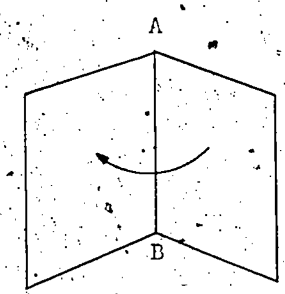


Figure 14

In this case, the line $\overleftrightarrow{AB}$ is called the <u>flip axis</u>. Consider the congruent pair of figures, E and F, shown below.

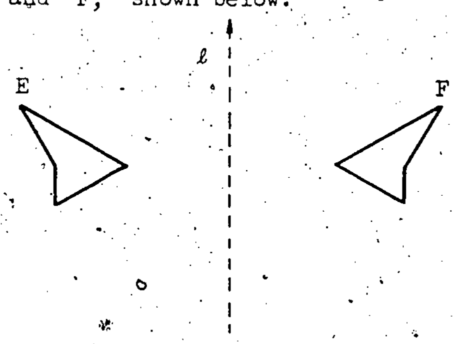


Figure 15

Figure E may be made to coincide with figure F by a flip about ℓ as the flip axis. We say that figure F is the <u>flip image</u> of figure E or that the flip maps figure E onto figure F.

In figure 16, note how the figures are related by flips about n and m.

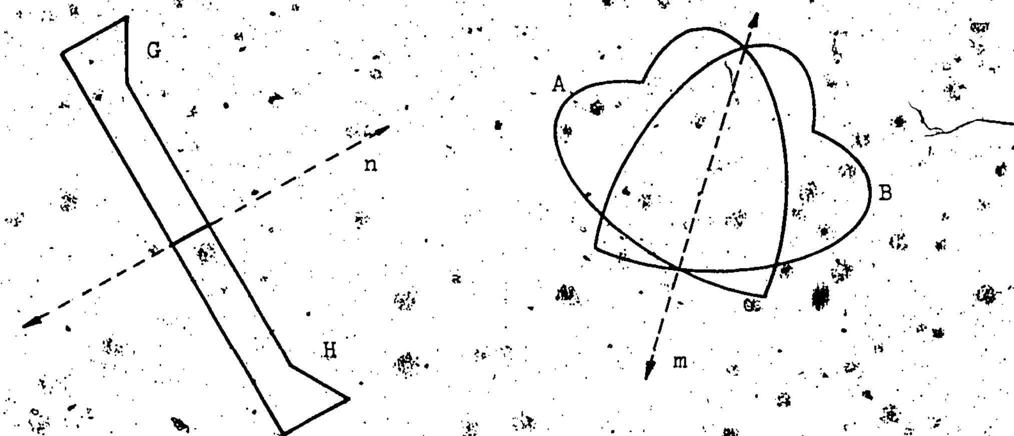


Figure 16

The line that is chosen for the flip can make a lot of difference. Consider the same figure but two different lines. What happens that is interesting in the second case below?

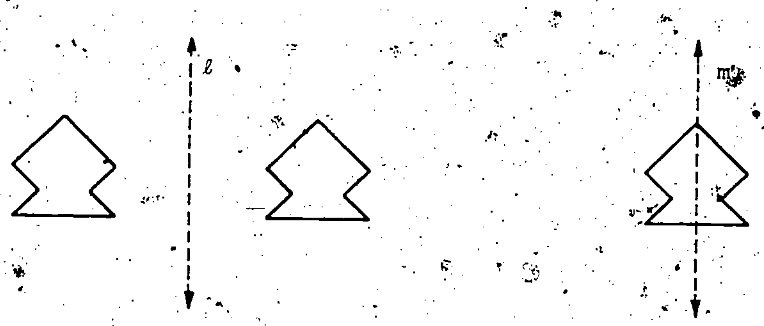


Figure 17

If we flip this figure on m as the flip axis we have exactly the same set of points that we had before the motion took place. That is, the figure is its own image for the flip described. In such a case, we say that the figure is <u>invariant</u> under the motion.

It is interesting to observe that sometimes two congruent figures can be made to coincide by different motions. For example, figure H can be made

to coincide with figure G by a flip or by a turn/slide or by other ways. (See figures 18 and 19.) Find one other way.

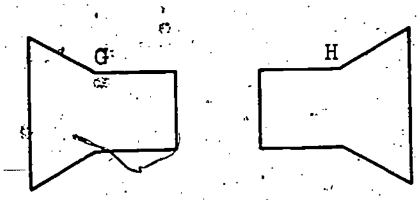


Figure 18

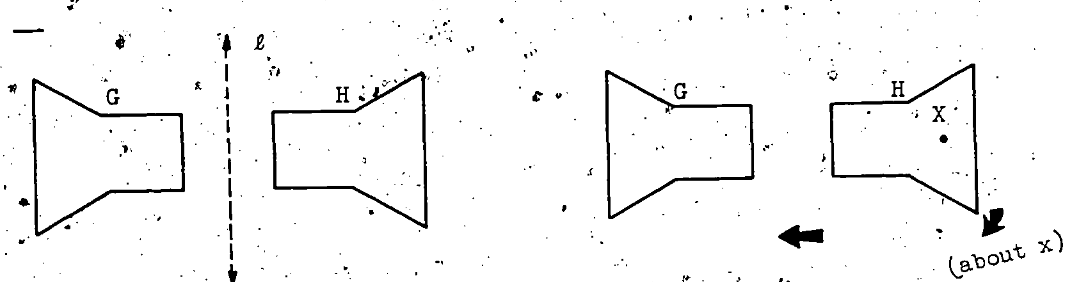
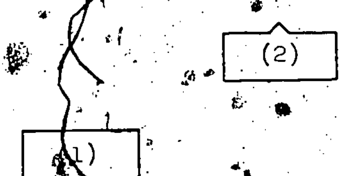


Figure 19

Exercises 8-8

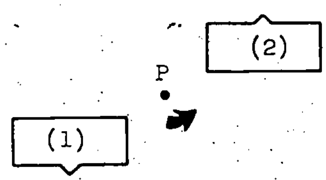
1. In each case, the given pair of figures are congruent. For each congruent pair, show exactly how the figure marked (1) may be mapped into the figure marked (2) by using the motions of slide, turn, or flip, or a combination of these methods. Use arrows to describe slides or turns and dotted lines to identify flip axes. Describe more than one method of making the figures coincide if you can.

Example:



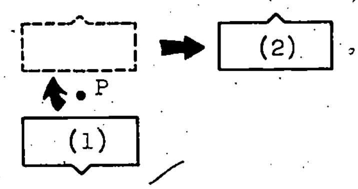
71

Possible Solution:



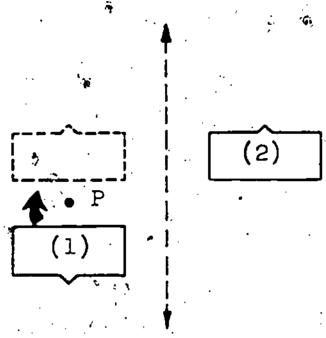
rotation about point P

Possible Solution:



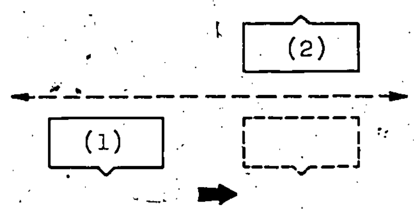
rotation/slide

Possible Solution:



rotation/flip

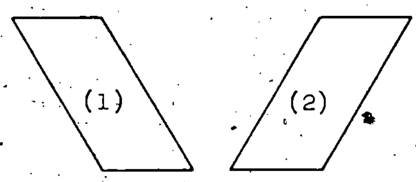
Possible Solution:



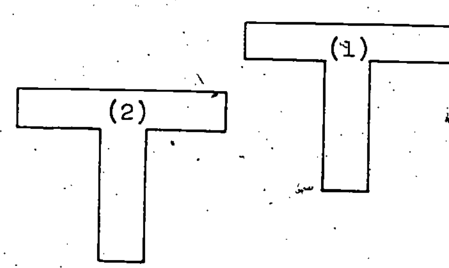
slide/flip

Can you find any others? Hint: there are some!

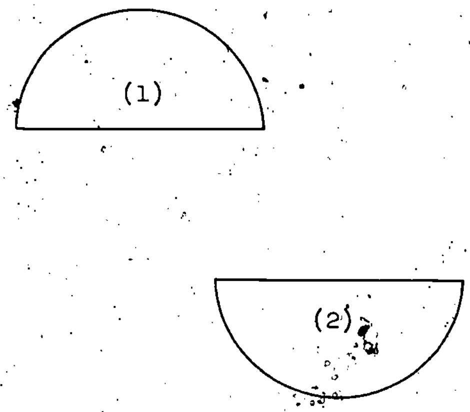
(a)



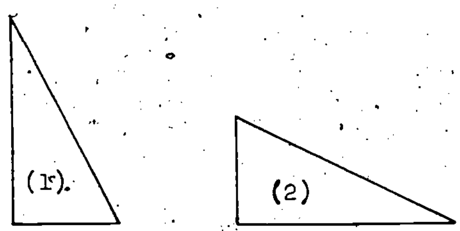
(b)



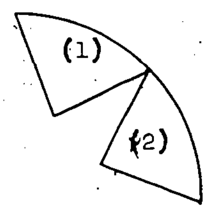
(c)



(d)

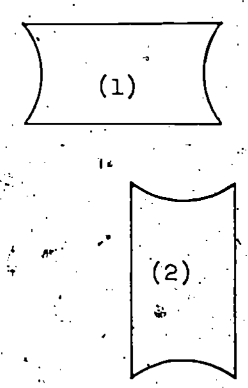


(e)



(f)

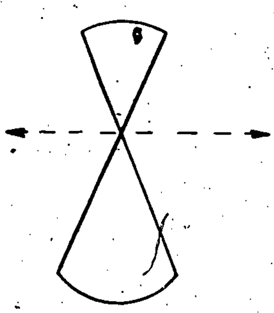
(g)



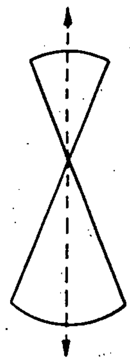
Can you plan a problem like those above that will challenge your classmates?

2. The figures below are to be flipped on the dotted lines as axes. In which cases is the figure invariant under the indicated motion?

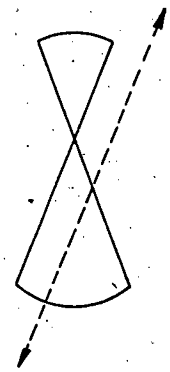
(a)



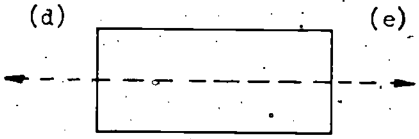
(b)



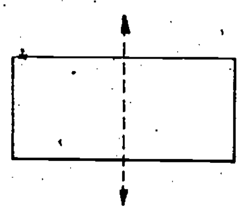
(c)



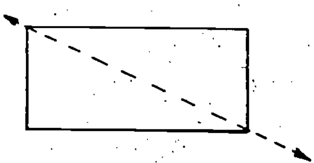
(d)



(e)

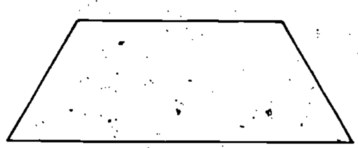


(f)



3. For each figure below, find all flip axes such that the figure is invariant.

(a)



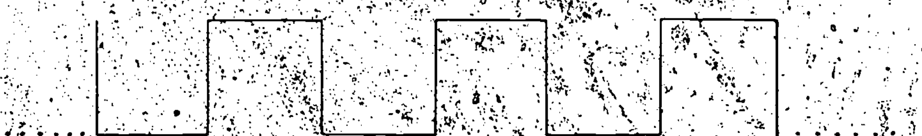
(b)

(c)  (d) 

Can you draw a figure that has interesting invariants?

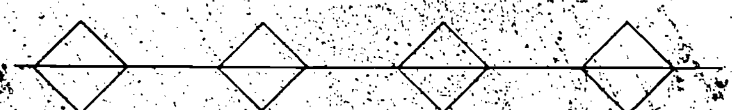
4. Can you find any figures having infinitely many flip axes?

5. <u>BRAINBUSTER</u>. Think of each figure below as a design on a wall and assume that it extends infinitely in both directions. Describe the motion or motions that will carry the design into itself. We say that each of these motions induces a congruence of the design with itself, or that the design is invariant under each of these motions.

(a) 

(b) 

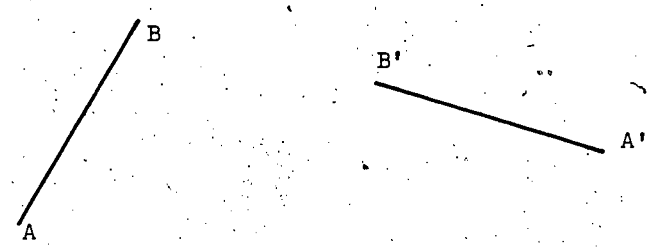
Describe two types of motion.

(c) 

Describe two types of motion. Can you find a third type of motion?

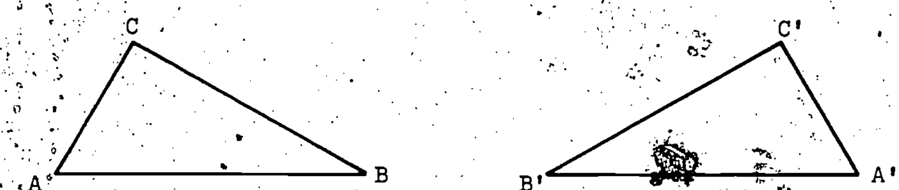
8-9. Orientation

$\overline{AB}$ is congruent to $\overline{A'B'}$.



We know from our meaning of the word "congruent" that $\overline{AB}$ (or a copy of it) could be made to fit exactly on $\overline{A'B'}$. In fact, we could slide $\overline{AB}$ so that A coincides with A', and then we could turn $\overline{AB}$ about point A' (= A) until B coincides with B'.

Consider $\triangle ABC$ and $\triangle A'B'C'$.



$\triangle ABC \cong \triangle A'B'C'$. Try to make a copy of $\triangle ABC$ fit exactly on $\triangle A'B'C'$.

If you tried to fit a copy of $\triangle ABC$ on $\triangle A'B'C'$, you no doubt found that your copy had to be flipped over before it could be made to coincide with $\triangle A'B'C'$. No amount of sliding or turning of $\triangle ABC$ without flipping will make the two triangles fit together.

Something very simple, yet very important, is at work here. Triangles in the same plane have a property (not shared by segments in a plane) which we will call orientation. We say that $\triangle ABC$ and $\triangle A'B'C'$ have opposite orientation because even though they are congruent they can not be made to coincide by merely sliding or turning either one in the plane of the page of the book.

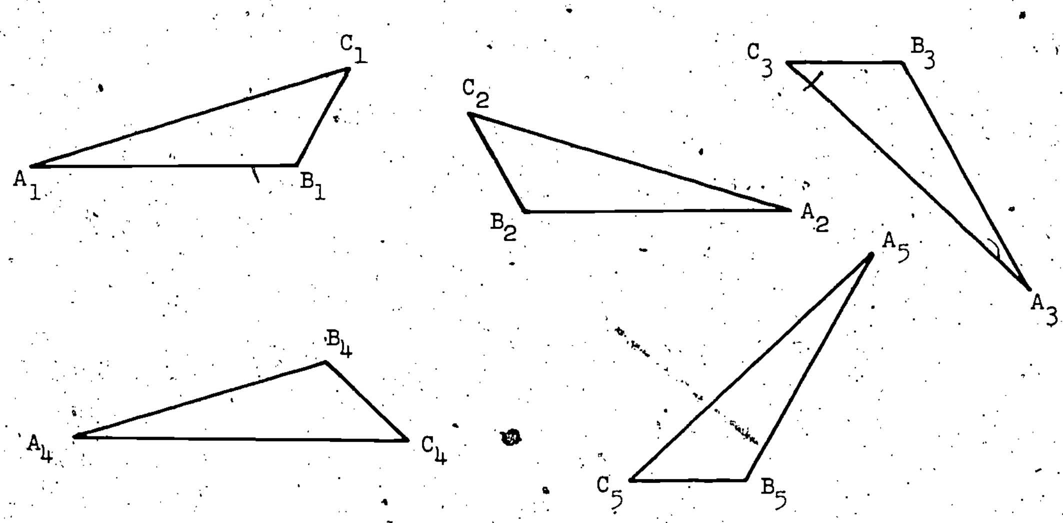
Definition: When we say two congruent figures in the same plane have the "same orientation", we mean they can be made to coincide without a flip.

We sometimes say that when we flip a figure in the plane we "reverse" its orientation.

Exercises 8-9a

(Class Discussion)

1. The triangles below are all congruent. Their vertices are named "A_1", "A_2", "A_3", etc. (read "A sub 1", "A sub 2", etc.) to show corresponding parts clearly.



Trace around each triangle, touching the vertices of each in the order A, B, C.

(a) List the triangles you were forced to trace clockwise.

(b) List the triangles you were forced to trace counterclockwise.

(c) List the triangles which do not need a flip to make them (or copies of them) coincide with $\Delta A_5B_5C_5$. (Include $\Delta A_5B_5C_5$ itself.)

(d) List the triangles which have opposite orientation to $\Delta A_5B_5C_5$. (Would you include $\Delta A_5B_5C_5$ itself?)

(e) Compare your lists in (a), (b), (c), and (d). Which lists are the same?

2. Draw any five-sided figure and name its vertices P, Q, R, S, and T. Then draw an exact copy of your figure as it would look after a flip and name its vertices P', Q', R', S', and T', showing the corresponding parts.

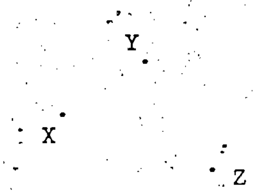
(a) Explain why PQRST and P'Q'R'S'T' have opposite orientation.

(b) Trace around each figure in alphabetical order. What do you notice about the directions of the two traces?

(c) Compare your results with your classmates'. What conclusion holds for all?

Exercises 8-9a suggest another way of characterizing orientation for figures in the same plane: two congruent figures have the same orientation if the traces through their corresponding points are in the same direction (clockwise or counterclockwise). If the traces go in opposite directions (one clockwise and one counterclockwise), then the congruent figures have opposite orientation.

In this way, we may talk about the idea of orientation in other situations. For example, consider the points X, Y, and Z.



The order X,Y,Z may be described as clockwise, and the order X,Z,Y may be described as counterclockwise. The orderings X,Y,Z and X,Z,Y have opposite orientation.

Exercises 8-9b

1. (a) Locate three noncollinear points on your paper and label them D, E, and F.

(b) There are six different arrangements which can be found of the letters D, E, F. Two of these are E, D, F and D, F, E. Write out the six arrangements and find the orientation of each from your diagram. How many different orientations are there?

(c) Which arrangements have the same orientation and which have opposite orientations?

(d) Can you discover a method for telling which arrangements have the same orientation and which have different orientations?

2\. (a) Locate three noncollinear points on your paper and label them A, B, and C.

(b) Locate point C' in several different positions so that A, B, C and A, B, C' have the same orientation. How can you describe the position of C' if A, B, C and A, B, C' are to have the same orientation?

(c) Locate point C' in several different positions so that A, B, C and A, B, C' have opposite orientations. How can you describe the position of C' if A, B, C and A, B, C' are to have opposite orientations?

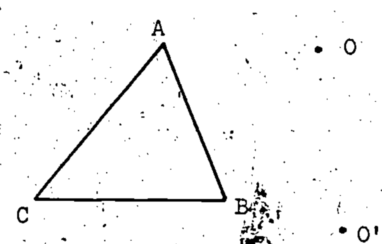
3\. Draw $\triangle ABC$ so that the order A,B,C has clockwise orientation. Mark some point O in the interior of $\triangle ABC$.

(a) What is the orientation of each of the following?

A,B,O
B,A,O
B,C,O
C,B,O
C,A,O
A,C,O

(b) Which of the orderings in part (a) have the same orientation as A,B,C?

4\. In the figure at the right, the order A,B,C has clockwise orientation. Points O and O' are in the exterior of $\triangle ABC$, and they do not lie on $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$ or $\overleftrightarrow{AC}$.



What is the orientation of each of the following?

A,B,O	A,B,O'
B,A,O	B,A,O'
B,C,O	B,C,O'
C,B,O	C,B,O'
C,A,O	C,A,O'
A,C,O	A,C,O'

5\. (a) Draw a line, $\overleftrightarrow{AB}$, on your paper.

(b) Locate points C and C' on the same side of $\overleftrightarrow{AB}$.

(c) Do A,B,C and A,B,C' have the same orientation or opposite orientation? Will your conclusion always apply if C and C' are on the same side of $\overleftrightarrow{AB}$?

6\. (a) Draw a line, $\overleftrightarrow{AB}$, on your paper.

(b) Locate points C and C' on opposite sides of $\overleftrightarrow{AB}$.

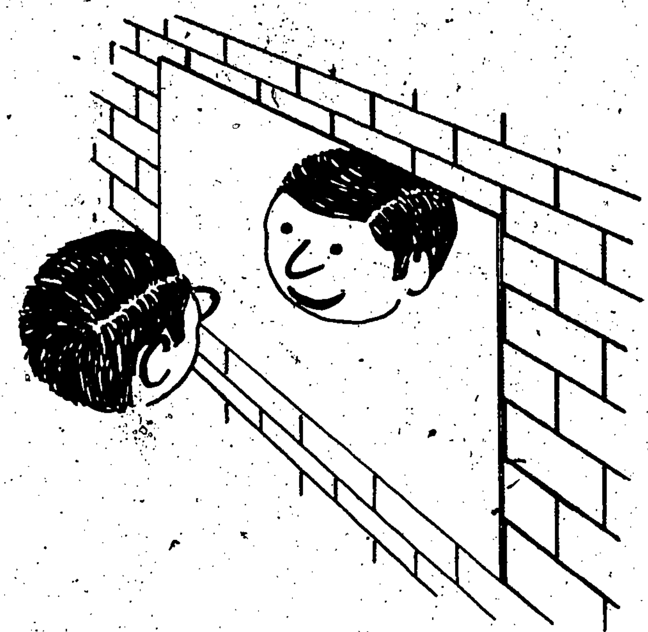
(c) Do A,B,C and A,B,C' have the same orientation or opposite orientation? Will your conclusion always apply if C and C' are on opposite sides of $\overleftrightarrow{AB}$?

7\. (a) Show how the property you observed in Exercise 6 explains the different results you obtained for points O and O' in Exercise 4.

(b) Now show how the property you observed in Exercises 5 and 6 explains your results in Exercise 3.

Your two hands are very close to being congruent figures in space. We could imagine a pair of hands which were so ideal that all their corresponding measures were equal. Even so, could we make these two ideal hands coincide? Imagine just the shape of the surface of the right hand; could it be made to fit exactly on the ideal left hand? Can a right hand glove fit exactly on a left hand?

It should be clear that we can think of figures in space that are congruent in all respects but which could not possibly be made to coincide even in our wildest imaginations.



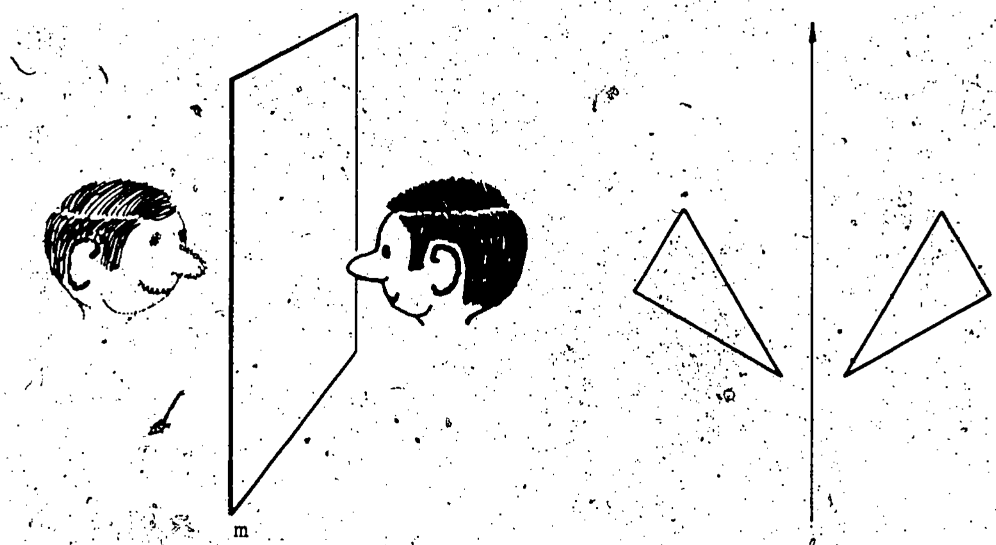
Perhaps the best example is your own image in a mirror. No matter how you twisted and turned you could never look exactly like it even though it is clearly an exact copy of you. (In our picture the fellow and his image in the mirror part their hair on opposite sides. There will always be some left and right differences.)

The image in the mirror is no different in size and shape. It merely has opposite orientation in space. Two congruent figures would have the same orientation in space if they could be made to fit exactly, one on another.

Look at the image of your left hand in a mirror. Does it have the same orientation as your real right hand? Your right and left hands are essentially congruent figures with opposite orientation. Reflecting a solid in a mirror changes its orientation.

It is interesting to think about the connections between flips in the plane and mirror reflections.

81



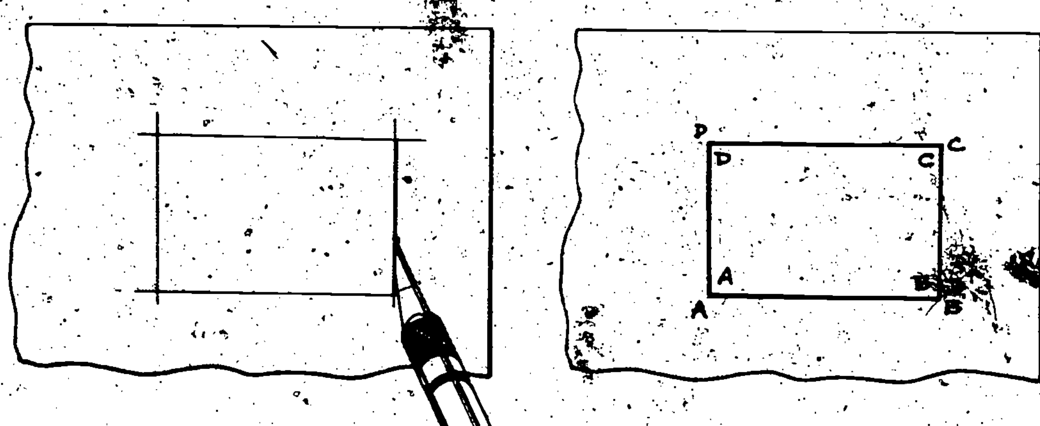
The flip axis ℓ acts so much like a mirror that mathematicians use the word "reflection" in place of our word "flip". You may have already noticed that our use of the word "image" (as in "flip image" and "turn image") probably comes from the common phrase "mirror image".

Exercise 8-9c

1. Does a mirror reverse left and right? (Be prepared to convince your classmates that your answer makes sense.)

8-10. Congruence of a Figure with Itself

Take a scrap of cardboard and cut a rectangle (not a square) out of it, as shown below.



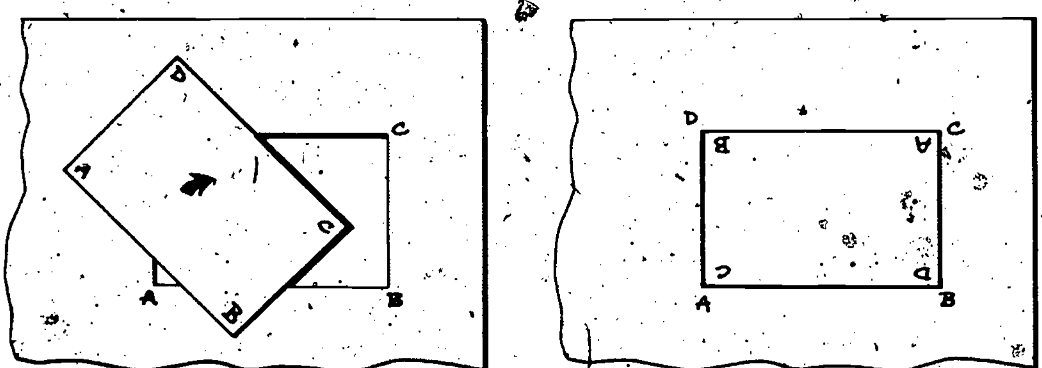
We label the vertices of the rectangle and the corners of the rectangular cut-out with capital letters.

If we take out the rectangular cut-out, in how many ways may it be fitted back into place?

1. Of course, we can fit it back in the exact position from which it was cut out. This fitting indicates the identity congruence

$$ABCD \longleftrightarrow ABCD.$$

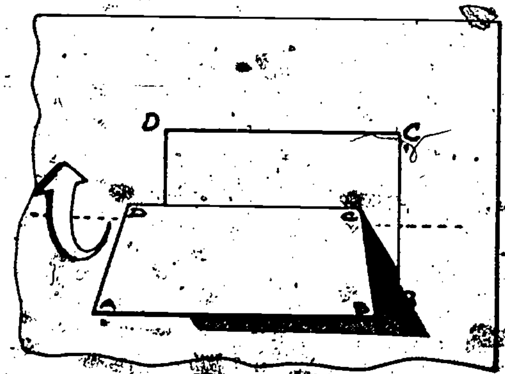
2. However, the rectangular cut-out may be fitted into the vacant space in other ways. Suppose we rotate the cut-out rectangle so that the corner A takes a position at C, the corner B takes a position at D, the corner C takes a position at A, and the corner D takes a position at B. We may describe this movement as a half-turn. The following fitting may now be made.

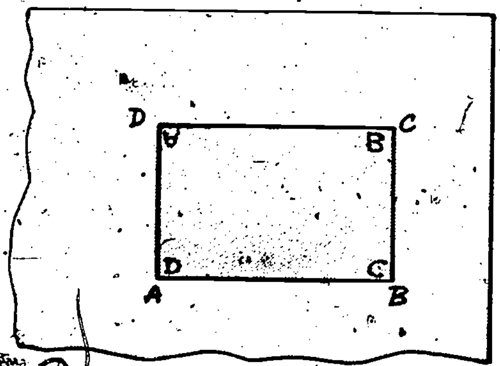


This fitting can be described as a congruence indicated by the following correspondence.

$$ABCD \longleftrightarrow CDAB$$

3. As you might have surmised we may take the rectangular cut-out in its original position and flip it over a horizontal axis. In this case, corner A takes a position at D, corner B takes a position at C, corner C takes a position at B, and corner D takes a position at A. The following fitting may now be made.

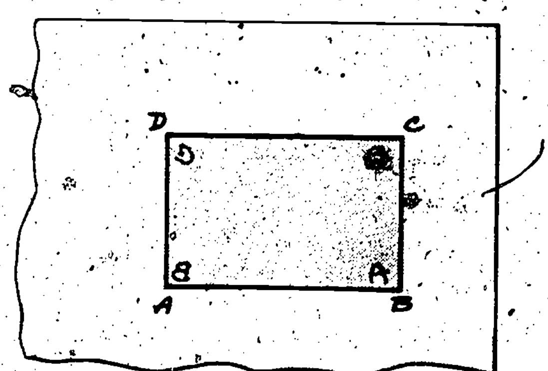




This fitting can be described as a congruence indicated by the following correspondence.

$$ABCD \longrightarrow DCBA$$

4. We may now rotate the flipped rectangular cut-out so that corner A takes a position at B, corner B takes a position at A, corner C takes a position at D, and corner D takes a position at C. The following fitting may now be made.



This fitting may be described as a congruence indicated by the following correspondence.

$$ABCD \longrightarrow BADC$$

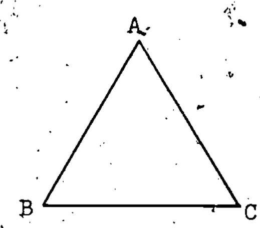
Do you think that we have identified all the possible ways in which the rectangle cut-out may be fitted back into place? In other words, in how many ways may a rectangle (not a square) be congruent to itself? Give a reason for your answer.

Exercises 8-10

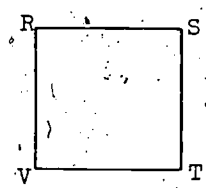
In the problems below indicate which correspondences are obtained by a rotation in the plane, which by reflection, and which by a combination of the two motions.

1. If ABCD is an isosceles trapezoid with $\overline{AD} \cong \overline{BC}$, write two correspondences that indicate that ABCD is congruent to itself.

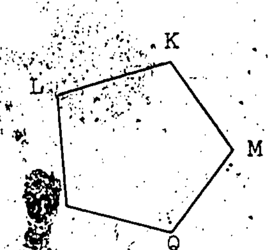
2. If $\triangle ABC$ is equilateral write the correspondences that indicate that $\triangle ABC$ is congruent to itself. There are six such correspondences.



3. If RSTV is a square write the correspondences that indicate that RSTV is congruent to itself. There are eight such correspondences.



4. KLPQM is a regular pentagon. Recall that in a regular pentagon the sides are congruent to each other and the angles are congruent to each other. Write the correspondences that indicate that KLPQM is congruent to itself. Did you get ten such correspondences that are distinct?



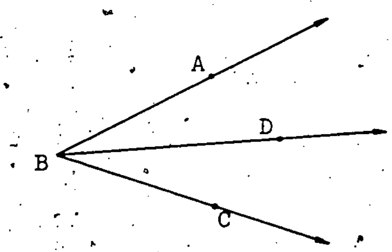
5. If a regular polygon has n sides how many correspondences are there which indicate that the regular polygon is congruent to itself?

8-11. Bisectors and Perpendiculars

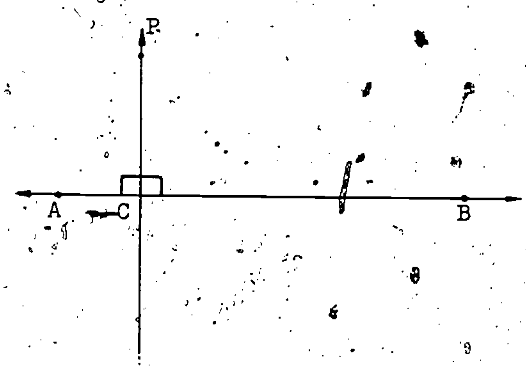
We have been using the ideas of midpoint of a segment and bisector of an angle informally, we will now define them precisely.

Definition: A point B is called the midpoint of $\overline{AC}$ if B is between A and C and $AB = BC$.

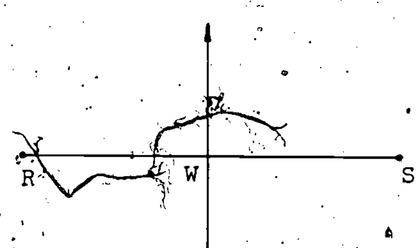
Recall that AB and BC are numbers. We say that point B bisects $\overline{AC}$ or $\overline{AC}$ is bisected at point B.



An angle may also be bisected. In the diagram at the right, $\overrightarrow{BD}$ bisects $\angle ABC$ because D is in the interior of $\angle ABC$ and $\angle ABD \cong \angle CBD$. The ray $\overrightarrow{BD}$ is called the bisector of $\angle ABC$. For convenience, we will also refer to any segment of $\overrightarrow{BD}$ with an endpoint at B as a bisector of $\angle ABC$.



In the figure at the right $\angle PCA \cong \angle PCB$. Since the sum of the measures of $\angle PCA$ and $\angle PCB$ is 180, the measures of $\angle PCA$ and $\angle PCB$ are each 90. That is, $\angle PCA$ and $\angle PCB$ are right angles. We say that $\overleftrightarrow{PC}$ is perpendicular to $\overleftrightarrow{AB}$. In symbols we write $\overleftrightarrow{PC} \perp \overleftrightarrow{AB}$ or $\overleftrightarrow{AB} \perp \overleftrightarrow{PC}$.

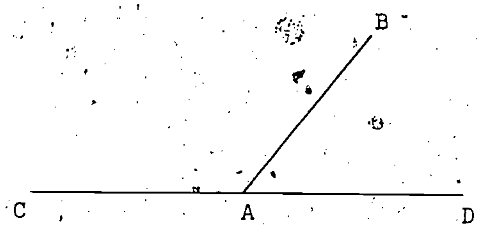


If $\overline{RW} \cong \overline{WS}$ and $\ell \perp \overleftrightarrow{RS}$ and ℓ contains W, we say that the line ℓ is the perpendicular bisector of $\overline{RS}$. For convenience, we will also refer to any segment or ray contained in line ℓ and containing the point W as a perpendicular bisector of $\overline{RS}$.

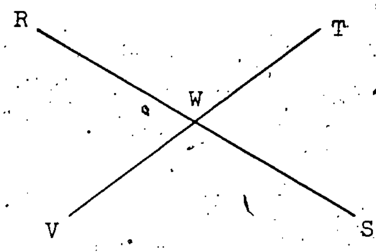
Exercises 8-11

1. In each case, select all pairs of congruent segments, or congruent angles.

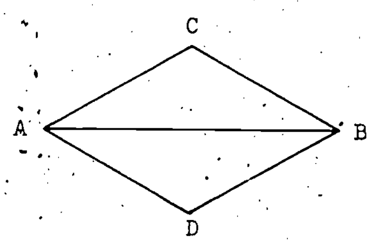
(a) $\overline{AB}$ passes through the midpoint of $\overline{CD}$.



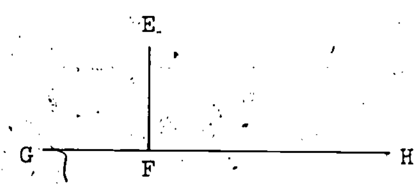
(b) $\overline{RS}$ and $\overline{TV}$ bisect each other.



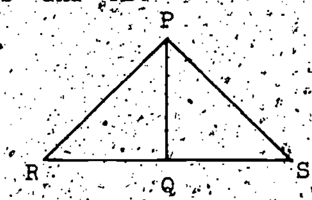
(c) $\overline{AB}$ bisects $\angle CAD$



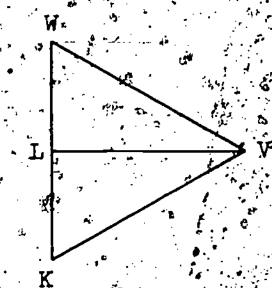
(d) $\overline{EF} \perp \overline{GH}$



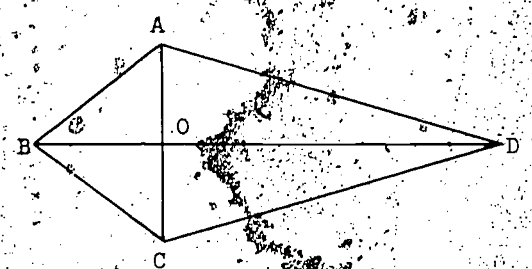
(e) $\overrightarrow{PQ}$ bisects $\angle RPS$ and $\overline{RS}$.



(f) $\overline{LV}$ is a perpendicular bisector of $\overline{WK}$.



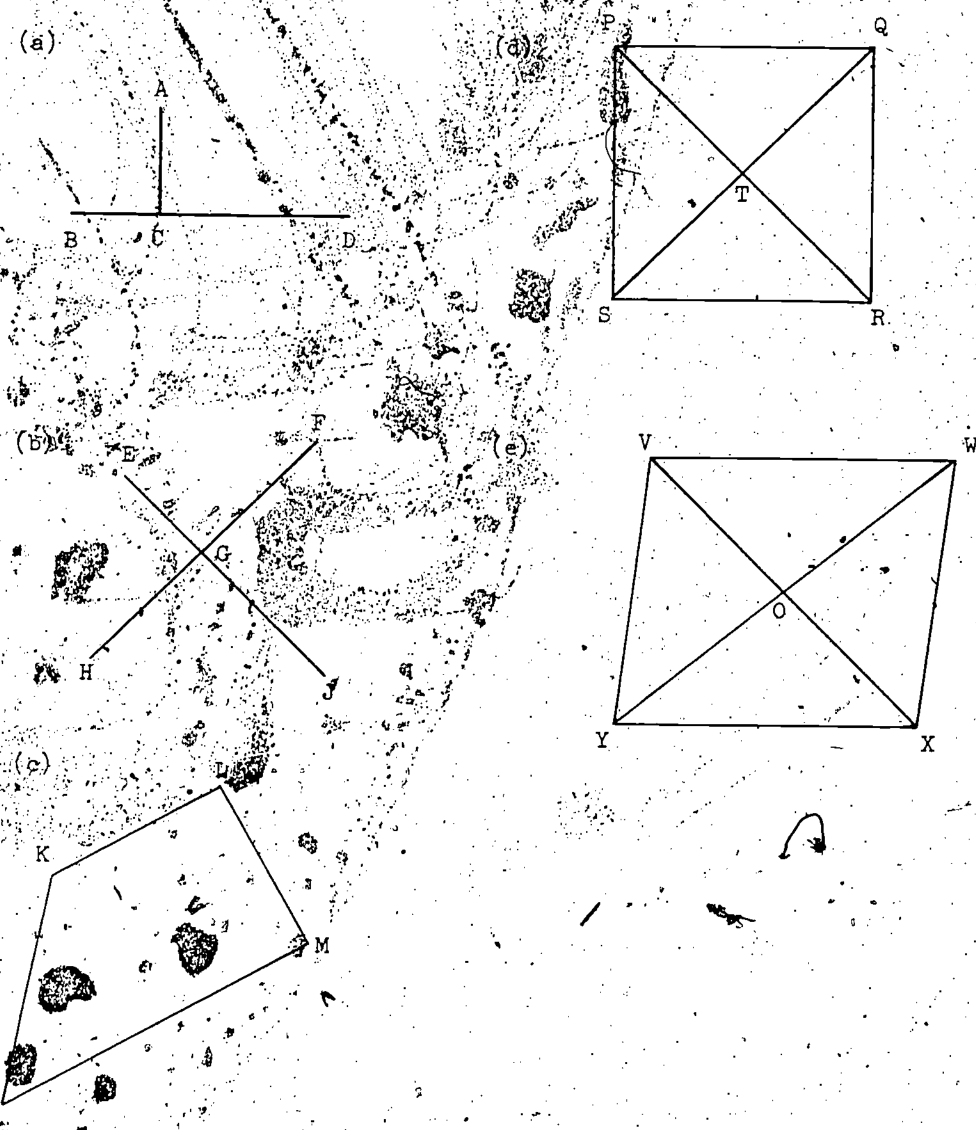
(g) $\overline{BD}$ is a perpendicular bisector of $\overline{AC}$.



2. In the figures below, if two lines appear to be perpendicular assume that they really are. Then:

Name the pairs of perpendicular segments.

Name the right angles.



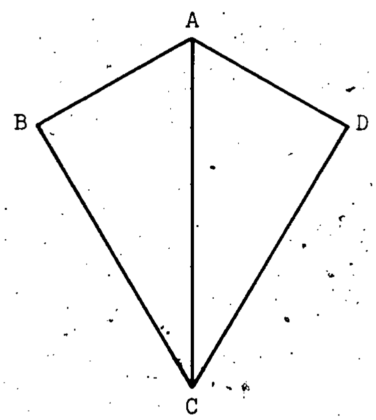
8-12. Writing a Proof

In this section we will study how to write a proof in an organized fashion. The ability to write a proof in an orderly manner is very helpful when we deal with harder problems; however, we will start with some simple problems to learn the method.

Exercises 8-12a

(Class Discussion)

1. Study the proof provided below.



Given: $\overline{AB} \cong \overline{AD}$

$\overline{CA}$ bisects $\angle BAD$

To prove: $\angle B \cong \angle D$

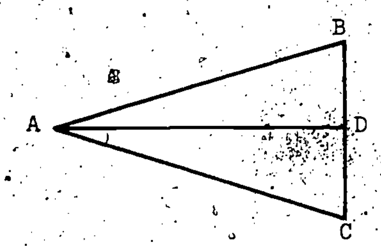
Plan: We know that two angles are congruent if they are corresponding angles of congruent triangles. Thus if we can prove that $\triangle BAC \cong \triangle DAC$ we will know that $\angle B \cong \angle D$.

Copy the diagram above and mark the corresponding congruent sides and angles which will show that $\triangle BAC$ and $\triangle DAC$ are congruent by the SAS property of congruence. The proof may be written as follows.

Proof

Statements	Reasons
1. $\overline{AB} \cong \overline{AD}$	1. Given
2. $\angle BAC \cong \angle DAC$	2. $\overline{CA}$ bisects $\angle BAD$
3. $\overline{AC} \cong \overline{AC}$	3. Every segment is congruent to itself.
4. $\triangle BAC \cong \triangle DAC$	4. SAS property of congruence
5. $\angle B \cong \angle D$	5. Corresponding angles of congruent triangles are congruent.

2. Write the missing reasons in the proof below.



Given: $\overline{AD}$ bisects $\angle BAC$

$\overline{AD} \perp \overline{BC}$

To prove: $\overline{AB} \cong \overline{AC}$

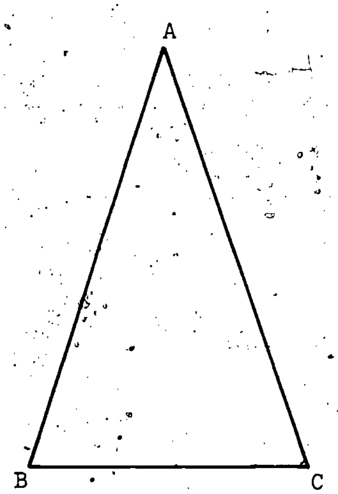
Plan: Prove $\triangle ABD \cong \triangle ACD$ by ASA.

Proof

Statements	Reasons
1. $\angle BAD \cong \angle CAD$	1. Why?
2. $\overline{AD} \cong \overline{AD}$	2. Why?
3. $\angle BDA$ and $\angle CDA$ are right angles.	3. Perpendicular segments form right angles.
4. $\angle BDA \cong \angle CDA$	4. Why?
5. $\triangle ABD \cong \triangle ACD$	5. Why?
6. $\overline{AB} \cong \overline{AC}$	6. Why?

3. In the following theorem, supply the missing reasons in the written proof.

<u>Isosceles Triangle Theorem 1</u>: If two sides of a triangle are congruent, then the angles opposite those sides are congruent. (This statement is called a <u>theorem</u> because we prove it using properties which we have already found to be true. Both this statement and the one in Exercise 4 which follows, will be very useful in helping us to discover and establish many new results.)



Given: Isosceles triangle ABC with $\overline{AB} \cong \overline{AC}$.

To prove: $\angle B \cong \angle C$.

Plan: In Exercises 8-10 we found correspondences of figures which indicated that the figures were congruent with themselves. We look for such a correspondence for an isosceles triangle. If we set up the correspondence

$A \longleftrightarrow A$

$B \longleftrightarrow C$

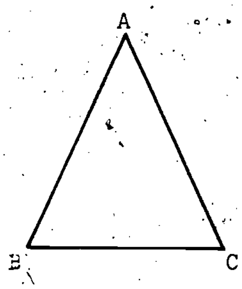
$C \longleftrightarrow B$

between the vertices of $\triangle ABC$, then $\triangle ABC \cong \triangle ACB$ and $\angle B \cong \angle C$ since they are corresponding parts of congruent triangles. The written proof is as follows.

Proof

Statements	Reasons
1. $\overline{AB} \cong \overline{AC}$ $\overline{AC} \cong \overline{AB}$	1. Why?
2. $\overline{BC} \cong \overline{BC}$	2. Why?
3. $\triangle ABC \cong \triangle ACB$	3. SSS property of congruence
4. $\angle B \cong \angle C$	4. Why?

4. Complete the following written proof.



Isosceles Triangle Theorem 2: If two angles of a triangle are congruent, then the sides opposite those angles are congruent.

Given: $\triangle ABC$ with $\angle B \cong \angle C$

To prove: $\overline{AB} \cong \overline{AC}$ and therefore $\triangle ABC$ is isosceles

Plan: Prove that $\triangle ABC \cong \triangle ACB$ by the ASA property.

Proof

Statements	Reasons
?	?

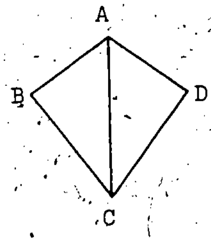
Exercises 8-12b

In each case, copy the proof and supply the missing reasons.

1. Given: $\overline{AB} \cong \overline{AD}$

   $\overline{BC} \cong \overline{DC}$

   To prove: $\angle B \cong \angle D$



Proof

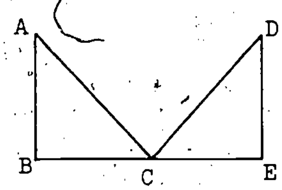
Statements	Reasons
1. $\overline{AB} \cong \overline{AD}$	1. ________
2. $\overline{BC} \cong \overline{CD}$	2. ________
3. $\overline{AC} \cong \overline{AC}$	3. ________
4. $\triangle ABC \cong \triangle ADC$	4. ________
5. $\angle B \cong \angle D$	5. ________

2. Given: $\overline{AB} \cong \overline{DE}$

   $\overline{BC} \cong \overline{EC}$

   $\overline{AB} \perp \overline{BC}$, $\overline{DE} \perp \overline{BC}$

   To prove: $\overline{AC} \cong \overline{DC}$



Proof

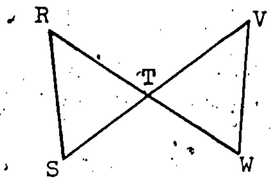
Statements	Reasons
1. $\overline{AB} \cong \overline{DE}$	1. ________
2. $\angle B$ and $\angle E$ are right angles	2. ________
3. $\angle B \cong \angle E$	3. ________
4. $\overline{BC} \cong \overline{EC}$	4. ________
5. $\triangle ABC \cong \triangle DEC$	5. ________
6. $\overline{AC} \cong \overline{DC}$	6. ________

93

3. Given: $\angle S \cong \angle W$

$\overline{ST} \cong \overline{WT}$

To prove: $\angle R \cong \angle V$



Proof

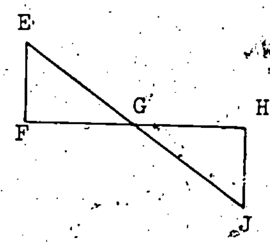
Statements	Reasons
1. $\angle S \cong \angle W$	1. ________
2. $\overline{ST} \cong \overline{WT}$	2. ________
3. $\angle RTS \cong \angle VTW$	3. ________
4. $\triangle TRS \cong \triangle TVW$	4. ________
5. $\angle R \cong \angle V$	5. ________

In each case copy the proof and supply the missing steps and reasons.

4. Given: $\overline{FG} \cong \overline{HG}$

$\overline{EF} \perp \overline{FH}$, $\overline{JH} \perp \overline{FH}$

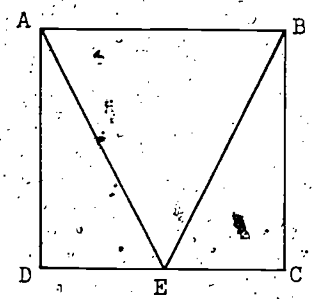
To prove: $\overline{EF} \cong \overline{JH}$



Proof

Statements	Reasons
1. $\angle EFG$ and $\angle JHG$ are right angles	1. ________
2. ________	2. All right angles are congruent.
3. $\angle EGF \cong \angle HGJ$	3. ________
4. ________	4. Given
5. $\triangle EGF \cong \triangle JGH$	5. ________
6. ________	6. ________

5. If ABCD is a square and E is the midpoint of $\overline{DC}$, then $\overline{AE} \cong \overline{BE}$.



Given: ABCD is a square; E is the midpoint of $\overline{DC}$.

To prove: $\overline{AE} \cong \overline{BE}$

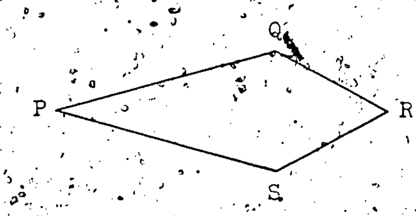
Proof

Statements	Reasons
1. $\overline{AD} \cong \overline{BC}$	1. ________
2. $\angle D$ and $\angle C$ are right angles.	2. ________
3. $\angle D \cong \angle C$	3. ________
4. $\overline{DE} \cong \overline{EC}$	4. ________
5. ________	5. ________
6. ________	6. ________

Give the complete proof of each of the following using problems 1 to 5 as models.

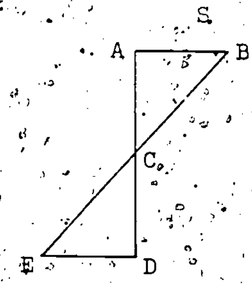
6. Given: $\overline{PQ} \cong \overline{PS}$
$\overline{QR} \cong \overline{SR}$

To prove: $\angle Q \cong \angle S$



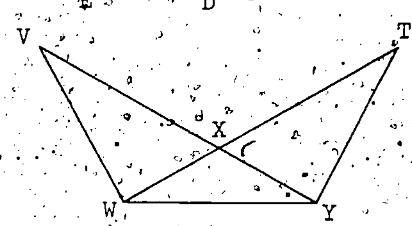
7. Given: $\overline{AC} \cong \overline{DC}$
$\overline{DA} \perp \overline{AB}$
$\overline{CD} \perp \overline{ED}$

To prove: $\overline{AB} \cong \overline{DE}$



8. Given: $\overline{VX} \cong \overline{TX}$
$\overline{WX} \cong \overline{YX}$

To prove: $\angle XWV \cong \angle XYT$

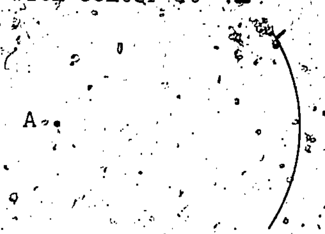


8-13. Construction of Rhombuses

The properties of congruence that we have discussed earlier in the chapter help us to discover facts about other geometrical figures. In this section, we will discover some properties of the rhombus. A rhombus is a quadrilateral (four-sided polygon in a plane) with all sides having equal lengths. We will see that these properties of the rhombus are both important and useful.

It is very easy to construct a rhombus using a straight edge and a compass, as shown below.

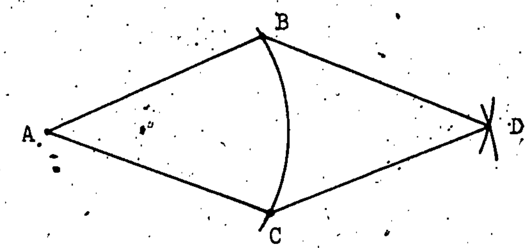
1. Select any point, A, as a vertex of the rhombus and draw an arc of a circle with center at A.



2. Since the sides of a rhombus have the same measure, two of the other vertices of the rhombus would lie on a circle with the center at A.

3. Again, since the sides of a rhombus have the same measure, we draw intersecting circular arcs with the same radius as the original circle.

4. Call this point of intersection D and draw $\overline{AB}$, $\overline{BD}$, $\overline{DC}$, and $\overline{CA}$.



Now, the four distances $m\overline{AB}$, $m\overline{AC}$, $m\overline{CD}$, and $m\overline{BD}$ are all equal to the radius used in drawing our circles. Thus $\overline{AB} \cong \overline{AC} \cong \overline{CD} \cong \overline{BD}$, and ABDC is a rhombus.

As you can see, it is very easy to construct a rhombus. It is just as easy to construct a rhombus satisfying various special conditions. You will do this in the next exercise set. However, one definition will be needed in some of these exercises. A <u>diagonal</u> of a rhombus is a segment joining a pair of opposite vertices. Thus, in the rhombus ABCD (see Figure 20), the diagonals are the segments $\overline{AC}$ and $\overline{BD}$, as shown in Figure 21.

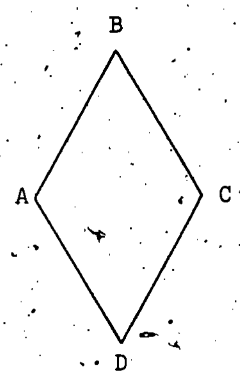


Figure 20

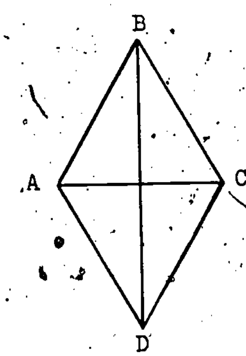


Figure 21

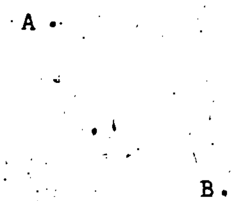
Exercises 8-13

(Class Discussion)

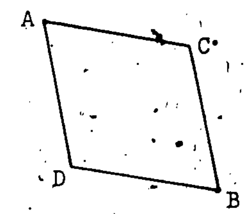
The construction of a rhombus above involved drawing three circular arcs with the same radius. In all the constructions below it will be helpful to keep in mind that two or three circular arcs with the same radius are drawn.

1. Construct a rhombus with two given points as opposite vertices. That is,

<u>Given</u>

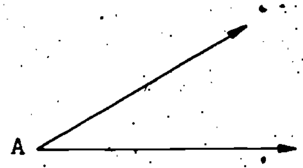


<u>Construct</u>

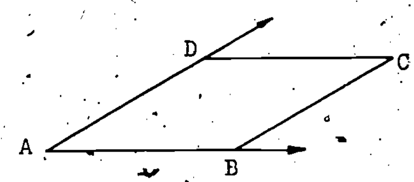


2. Construct a rhombus with two sides lying on a given angle. That is,

<u>Given</u>

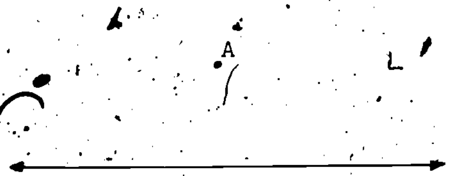


<u>Construct</u>

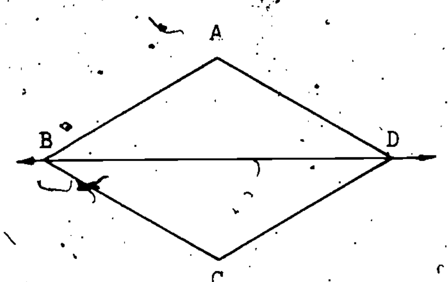


3. Construct a rhombus with one vertex at a given point and one diagonal lying on a given line. That is,

<u>Given</u>

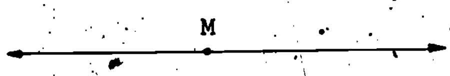


<u>Construct</u>

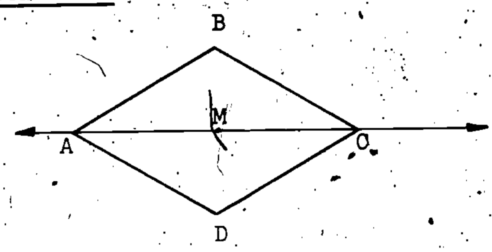


4. Construct a rhombus with one diagonal on a given line and the midpoint of that diagonal at a given point on that line. That is,

Given

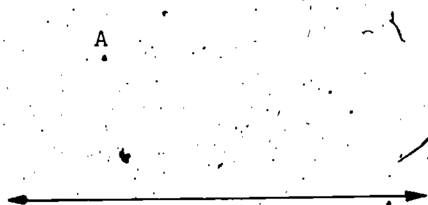


Construct

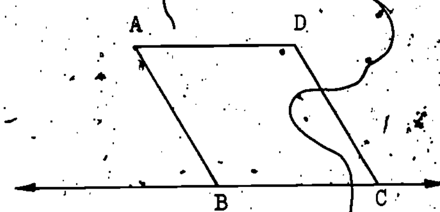


5. Construct a rhombus with one vertex at a given point and one side on a given line. That is,

Given



Construct



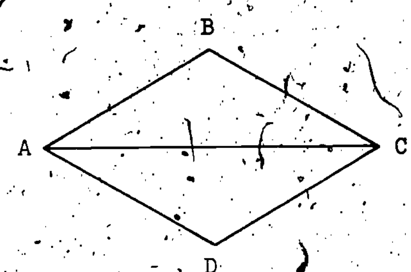
We are now ready to consider and prove the following important property of a rhombus:

A diagonal of a rhombus bisects the angle of the rhombus at each of its endpoints.

Given: ABCD is a rhombus

To prove: $\angle BAC \cong \angle DAC$
$\angle BCA \cong \angle DCA$

Plan: Use SSS to prove $\triangle ABC \cong \triangle ADC$ and then the desired angles are corresponding parts of congruent triangles.

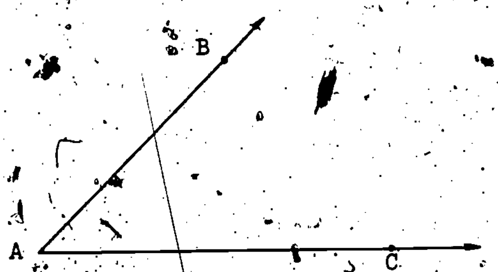


99

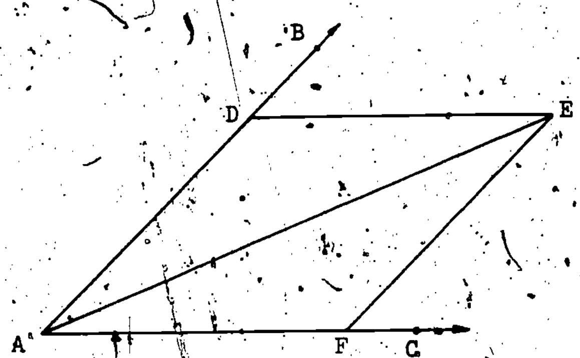
Proof

Statements	Reasons
1. $\overline{AB} \cong \overline{AD}$	1. All sides of a rhombus are congruent.
2. $\overline{BC} \cong \overline{DC}$	2. All sides of a rhombus are congruent.
3. $\overline{AC} \cong \overline{AC}$	3. A segment is congruent to itself.
4. $\triangle BAC \cong \triangle DAC$	4. SSS property of congruence.
5. $\angle BAC \cong \angle DAC$ $\angle BCA \cong \angle DCA$	5. Corresponding parts of congruent triangles are congruent.

This theorem enables us to make a geometrical construction for bisecting an angle. Consider $\angle BAC$, shown below:

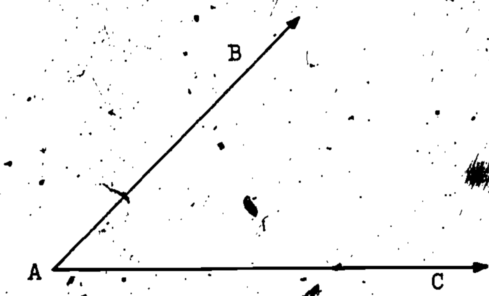


In problem 2 of the preceding exercise set we saw how to construct a rhombus with two sides on the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

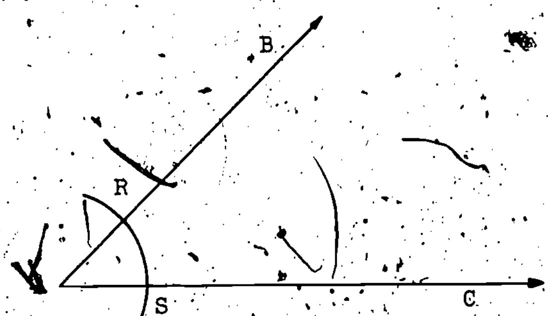


If we perform this construction then the above theorem tells that the diagonal $\overline{AE}$ of this rhombus bisects $\angle BAC$.

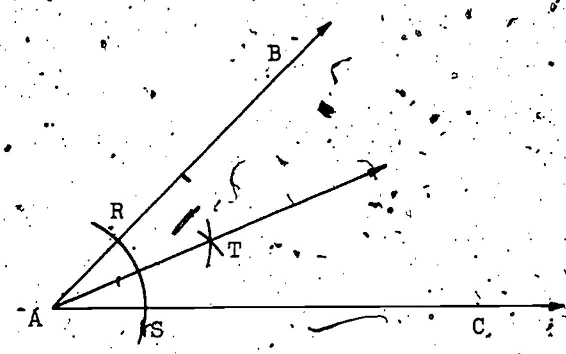
In actual practice, we do not actually draw the complete rhombus. We proceed as follows:



With A as a center we draw a circular arc cutting rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ at R and S, using any convenient radius.



With the same radius and points R and S as centers we draw intersecting circular arcs, like this:



Then we draw the diagonal $\overline{AT}$ to bisect $\angle BAC$. It is unnecessary to draw the other two sides of the rhombus, $\overline{RT}$ and $\overline{ST}$.

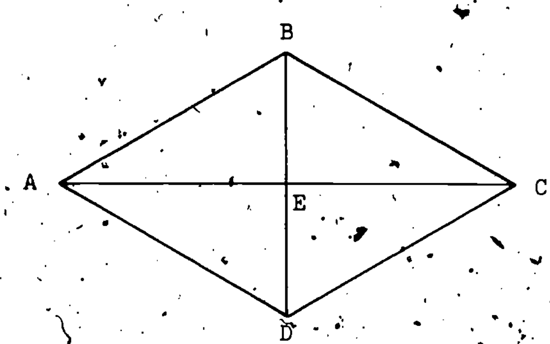
8-14. A Useful Property of the Rhombus

In the previous section we have seen how a property of the rhombus enabled us to devise a simple method of bisecting an angle. In this section we will become acquainted with another property of the rhombus which will have even richer applications.

The diagonals of a rhombus are perpendicular bisectors of each other.

Given: ABCD is a rhombus

To prove: $\overline{BE} \cong \overline{ED}$
$\overline{AE} \cong \overline{CE}$
$\overline{BD} \perp \overline{AC}$

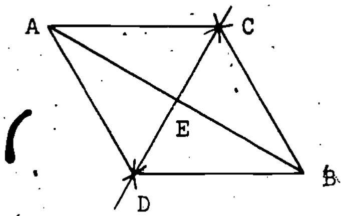


Proof

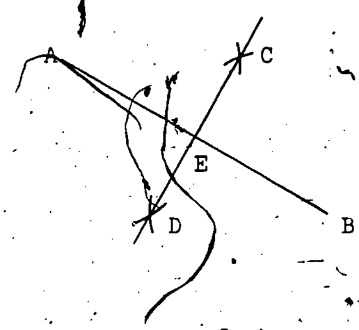
Statements	Reasons
1. $\overline{AB} \cong \overline{BC}$	1. All sides of a rhombus are congruent.
2. $\overline{BE} \cong \overline{BE}$	2. A segment is congruent to itself.
3. $\angle ABE \cong \angle CBE$	3. A diagonal of a rhombus bisects the angle of the rhombus at each of its endpoints.
4. $\triangle ABE \cong \triangle CBE$	4. SAS property of congruence.
5. $\overline{AE} \cong \overline{CE}$ $\angle AEB \cong \angle CED$	5. Corresponding parts of congruent triangles are congruent.
6. $m\angle AEB = 90$	6. $\angle AEB$ and $\angle CEB$ have the same measure and the sum of their measures is 180.
7. $\overline{BD} \perp \overline{AC}$	7. $\angle AEB$ is a right angle.

In order to prove that $\overline{BE} \cong \overline{DE}$ we can show that $\triangle BAE \cong \triangle DAE$. This is left as an exercise for the student.

We may now use this property of rhombuses to make many constructions involving perpendiculars. For example, in problem 1 of the last exercise set you constructed a rhombus with opposite vertices at two given points A and B. We now see that the line through the other two vertices of this



rhombus is the perpendicular bisector of $\overline{AB}$, that is, $\overline{AE} \cong \overline{EB}$ and $\overleftrightarrow{CD} \perp \overline{AB}$. Is it necessary to draw the rhombus in order to perform this construction? Study the construction below and explain.



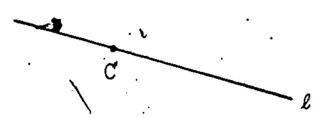
<u>Exercises 8-14</u>

(Class Discussion)

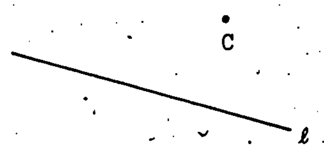
Devise methods for performing the constructions below. The solution of each of the problems involves one of the constructions already made in Exercises 8-13. For each problem identify which of the constructions in Exercises 8-13 should be used in solving the problem and explain why this construction gives the desired result.

1. Construct a line through a given point, C, which is perpendicular to a given line, ℓ. (C is not on ℓ.)

2. Same as (1) except that point C is on ℓ.



3. C is a point not on the line ℓ. Construct the reflection of C in ℓ. That is, construct a point E so that $\overline{CE}$ is perpendicular to ℓ and so that ℓ bisects $\overline{CE}$. Why is E called the reflection of C in ℓ?



8.15. A Shortest Path Problem

One day after school some boys wanted to play baseball but they didn't have enough players so they invented a new version of the game. They used two boxes and a straight line, as shown in Figure 22.

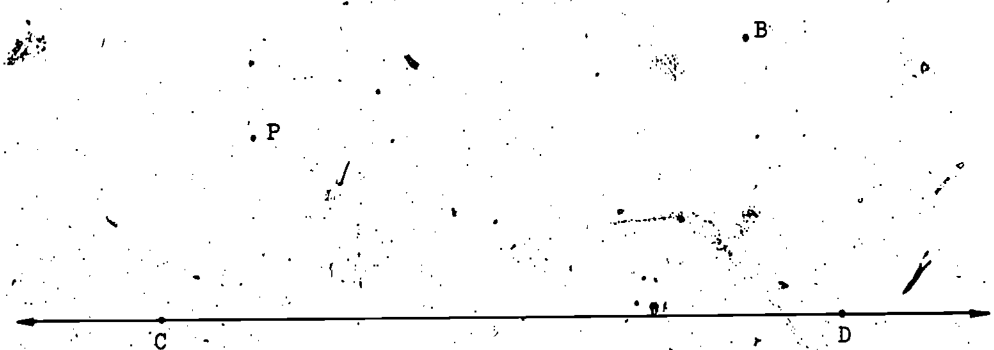
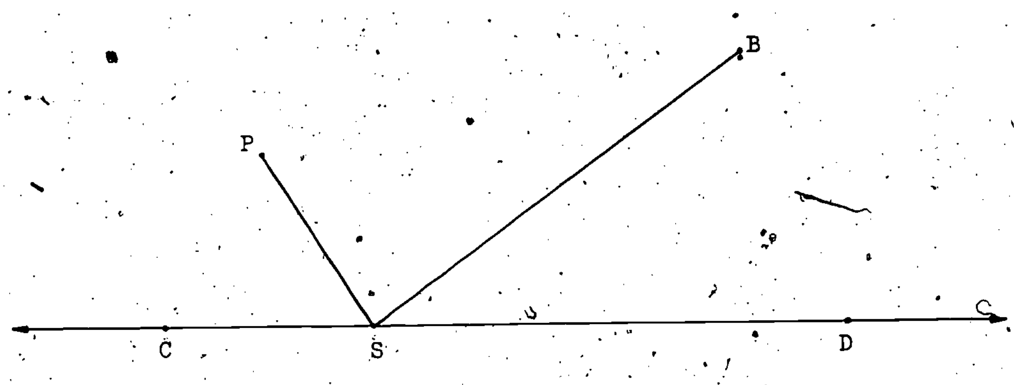


Figure 22

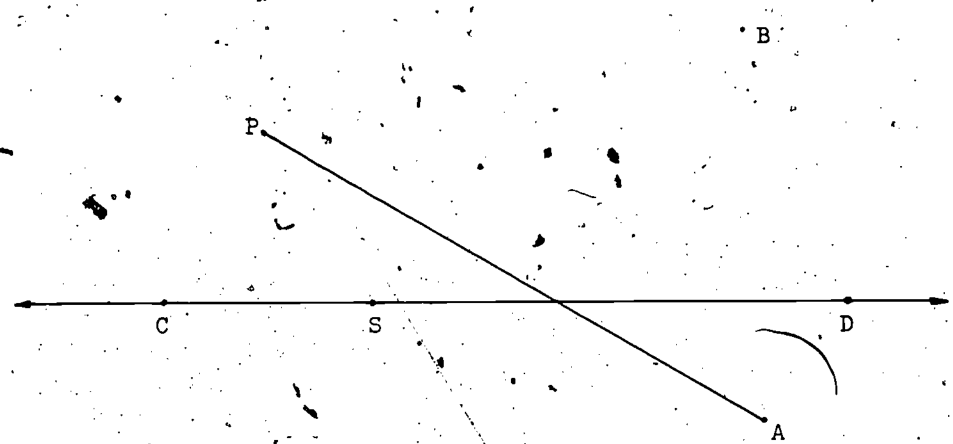
The batter stood at the plate, P, and hit a pitched ball. Then he had to run from the plate P to the base B, touching the line $\overleftrightarrow{CD}$ on the way. He scored a run if he arrived at B before the opposing team could "force" him by returning the ball to the base B.

After the batter hit the ball, he would run in a straight line path to some point S on $\overleftrightarrow{CD}$ and then in a straight line path from S to B. There are many ways of doing this depending on the choice of the point S. One way is shown below.



Naturally, the runner wishes to choose the point S so as to obtain a path as short as possible. But how is he to do this? Is there a way in which we can use our knowledge of geometry to locate the point S which yields the shortest path?

First, let us note that the difficulty in the problem occurs because P and B are on the same side of the line $\overleftrightarrow{CD}$. Consider, for example, the corresponding problem of going from P to A touching the line $\overleftrightarrow{CD}$.



In this case, segment $\overline{PA}$ constitutes the solution of the problem since, of necessity, it crosses $\overleftrightarrow{CD}$. The point at which the runner touches $\overleftrightarrow{CD}$ is then the point of intersection of $\overleftrightarrow{CD}$ and $\overleftrightarrow{PA}$. But can we use this simple idea to solve the original problem? The fact is that we can!

First of all, find the reflection, B', of the point B in the line $\overleftrightarrow{CD}$, as shown in Figure 23.

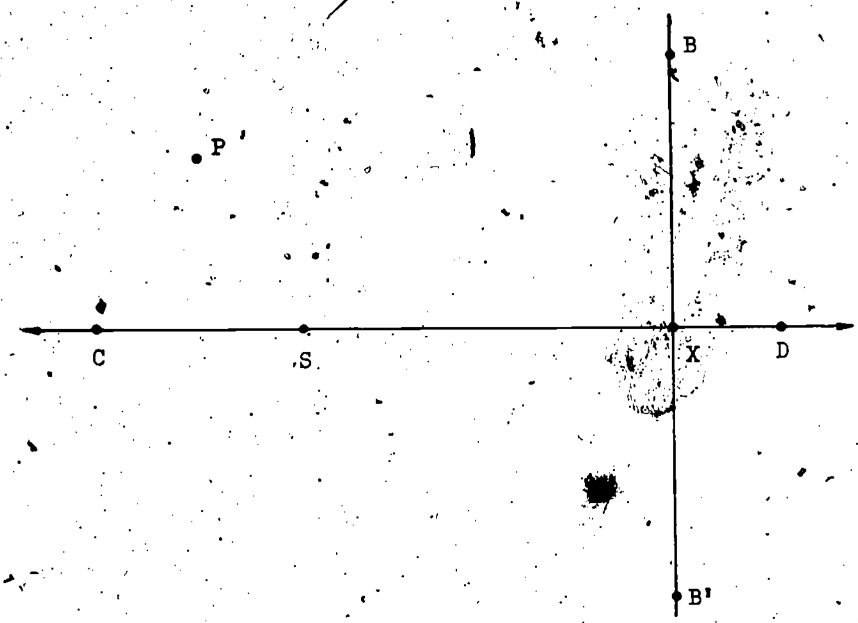


Figure 23

That is, $\overleftrightarrow{BB'} \perp \overleftrightarrow{CD}$ and $\overline{BX} \cong \overline{B'X}$. We learned how to make this construction in our work on the rhombus. Now select any point S on CD and draw the segments $\overline{PS}$, $\overline{SB}$ and $\overline{SB'}$. By use of congruent triangles we can show that $\overline{SB} \cong \overline{SB'}$. Which are the congruent triangles in this figure?

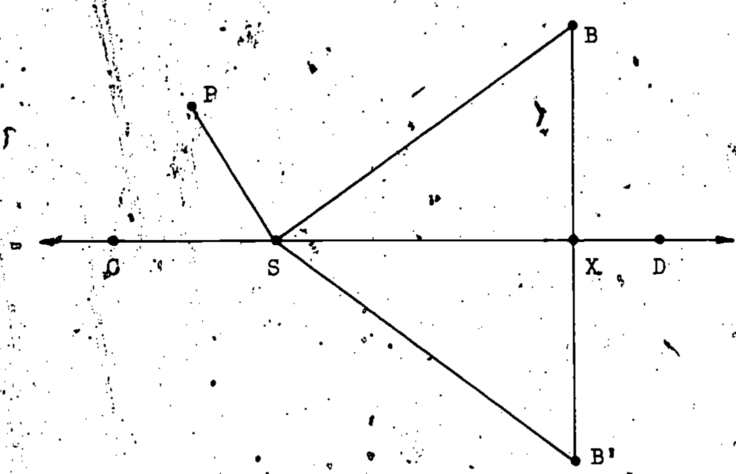


Figure 24

Thus the length of the path from P to S to B, which is $m\overline{PS} + m\overline{SB}$, is the same as the length of the path from P to S to B', which is $m\overline{PS} + m\overline{SB'}$. Thus the choice of S that yields the shortest path from P to ~~S~~ to B is the same as the point S that gives the shortest path from P to S to B'. But we have already seen which choice of S yields the shortest path from P to S to B'. What is it?

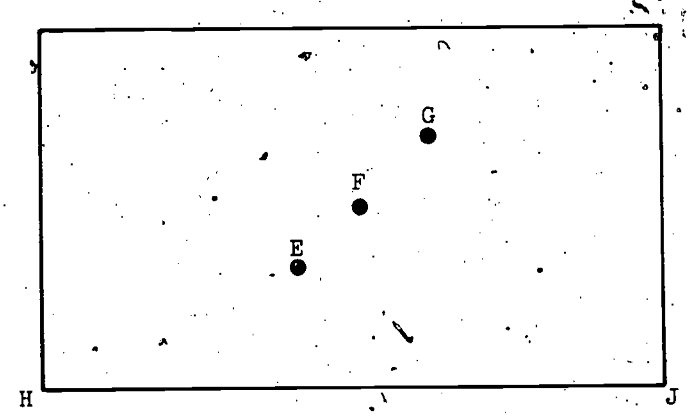
The problem presented here is merely one of a large class of mathematical problems dealing with the shortest path satisfying certain conditions. A number of such problems can be solved using the sort of "reflection principle" just employed.

Exercises 8-15

1. Express in writing the congruence which holds between the triangles of Figure 24. Explain why these triangles are congruent. Explain why $\overline{SB} = \overline{SB'}$.

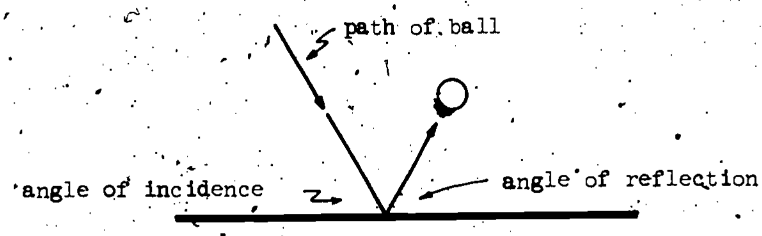
2. How do we locate the point S so that $m\overline{PS} + m\overline{SB}$ is as short as possible? Draw the completed figure locating the correct position of this point S and showing the shortest path from P to B which touches the line $\overleftrightarrow{CD}$.

3. On a billiard table three balls at points E, F, G lie in a straight line. We wish to strike the ball at E with our cue so that it will strike the ball at G without hitting the ball at F.



To do this it will be necessary to bounce the ball off an edge of the table, say edge $\overline{HJ}$. It is a principle of physics that when a

ball bounces off such an edge the "angle of reflection" is congruent to the "angle of incidence".



Give a geometrical construction to determine the path which the ball at point E must travel. [Hint: In problem 2 show that $\angle PSC \cong \angle BSD$.]

BRAINBUSTER. On a billiard table we wish to strike a ball at point E so that it will bounce off edges $\overline{HJ}$ and $\overline{JK}$ and then hit a ball at point F as shown

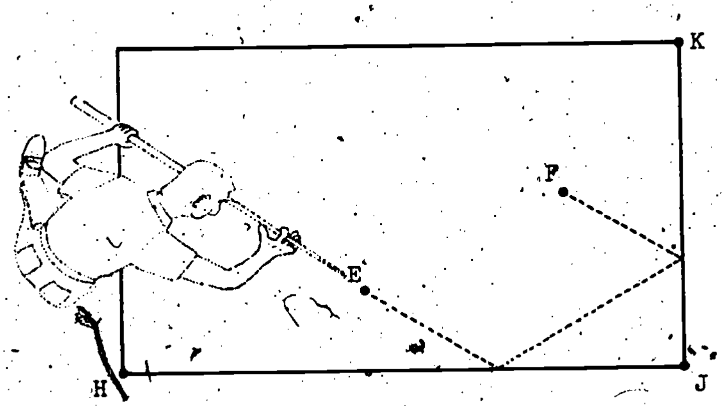


Figure 25

by the dashed line in Figure 25. (On each bounce the angle of incidence must be congruent to the angle of reflection.) Find a geometrical construction for the path that the ball at point E must travel. [Hint: Use the reflection principle twice.]

8-16. Summary

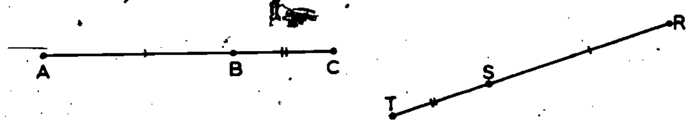
Section 8-2.

For any segments $\overline{AB}$ and $\overline{CD}$, if $m\overline{AB} = m\overline{CD}$ then $\overline{AB} \cong \overline{CD}$. Also if $\overline{AB} \cong \overline{CD}$ then $m\overline{AB} = m\overline{CD}$.

For any angles $\angle ABC$ and $\angle RST$, if $m\angle ABC = m\angle RST$ then $\angle ABC \cong \angle RST$. Also if $\angle ABC \cong \angle RST$ then $m\angle ABC = m\angle RST$.

Section 8-3.

Let B be between A and C, and S be between R and T. If $\overline{AB} \cong \overline{RS}$ and $\overline{BC} \cong \overline{ST}$, then $\overline{AC} \cong \overline{RT}$. If $\overline{AC} \cong \overline{RT}$, and $\overline{BC} \cong \overline{ST}$, then $\overline{AB} \cong \overline{RS}$.



Section 8-4.

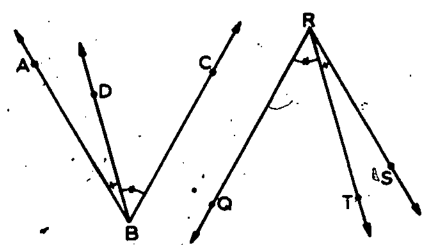
Two angles are called adjacent when

(a) they have a common vertex,

(b) they have a common ray or side, and

(c) their interiors do not overlap.

Let $\angle ABD$ and $\angle CBD$ be adjacent and let $\angle SRT$ and $\angle QRT$ be adjacent angles. (Points A, B, and C, and likewise points Q, R, and S are non-collinear.)



If $\angle ABD \cong \angle SRT$ and $\angle CBD \cong \angle QRT$ then $\angle ABC \cong \angle QRS$.
If $\angle ABC \cong \angle QRS$ and $\angle CBD \cong \angle QRT$ then $\angle ABD \cong \angle SRT$.

On a line $\overleftrightarrow{AB}$ if C is between A and B, and ray $\overrightarrow{CD}$ is not on the line, then $m\angle ACD + m\angle BCD = 180$.

The two pairs of non-adjacent angles formed by two intersecting lines are called vertical angles.

Any pair of vertical angles are congruent.

Section 8-5.

SAS Congruence Property:

If two sides and the included angle of one triangle have the same measures as two sides and the included angle of another triangle, then the two triangles are congruent.

ASA Congruence Property:

If two angles and the included side of one triangle have the same measures as two angles and the included side of another triangle, then the triangles are congruent.

SSS Congruence Property:

If three sides of one triangle have the same measures as the three sides of another triangle, then the triangles are congruent.

Section 8-6.

The corresponding vertices of two triangles are shown by a congruence statement as follows.

$\triangle ABC \cong \triangle KJL$

Corresponding Vertices

$A \longleftrightarrow K$

$B \longleftrightarrow J$

$C \longleftrightarrow L$

Thus a congruence statement gives us six facts at once, as shown below.

$\triangle ABC \cong \triangle KJL$

Corresponding Sides	Corresponding Angles
$\overline{AB} \cong \overline{KJ}$	$\angle A \cong \angle K$
$\overline{AC} \cong \overline{KL}$	$\angle B \cong \angle J$
$\overline{BC} \cong \overline{JL}$	$\angle C \cong \angle L$

Section 8-7.

A triangle which has at least two congruent sides is called an isosceles triangle.

A triangle whose three sides are congruent is called an equilateral triangle.

Section 8-8.

Two figures in the same plane that are congruent may be made to coincide by a motion.

There are three types of motion: slides, turns, and flips. A combination of these motions will describe any motion.

Section 8-9.

Two congruent figures in the same plane have the same orientation whenever they can be made to coincide without a flip.

Section 8-11.

A point B is called the midpoint of $\overline{AC}$ if B is between A and C, and $m\overline{AB} = m\overline{BC}$.

A ray $\overrightarrow{BD}$ bisects $\angle ABC$ if D is in the interior of $\angle ABC$, and $\angle ABD \cong \angle CBD$.

A line ℓ and a line m are perpendicular if they form a right angle.

Section 8-12.

Isosceles Triangle Theorems:

If two sides of a triangle are congruent, then the angles opposite those sides are congruent.

If two angles of a triangle are congruent, then the sides opposite those angles are congruent.

Section 8-13.

A diagonal of a rhombus bisects the angle of the rhombus at each of its endpoints.

Section 8-14.

The diagonals of a rhombus are perpendicular bisectors of each other.

Section 8-15.

For two points P and A on the same side of a line ℓ, the shortest path from P to A touching line ℓ may be found using the reflection principle.